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Stochastic Differential Systems with Memory: Theory, Examples and Applications (Sixth Workshop on Stochastic Analysis)

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STOCHASTIC DIFFERENTIAL SYSTEMS WITH MEMORY

THEORY, EXAMPLES AND APPLICATIONS

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Outline of Lectures

Lecture I. Existence.

1. Simple examples: The noisy feedback loop: $dx(t) = x(t-r)\,dW(t)$. Solution process $x(t)$ is not a Markov process in $\mathbf{R}$. No closed form solution when $r > 0$. Compare with the case $r = 0$. The logistic time-lag model with Gaussian noise:

$$dx(t) = [\alpha - \beta x(t-r)]x(t)\,dt + \sigma x(t)\,dW(t).$$

   The classical "heat-bath" model of R. Kubo: Motion of a large molecule in a viscous fluid.

2. General Formulation. Choice of state space. Pathwise existence and uniquenss of solutions to sfde's under local Lipschitz and linear growth hypotheses on the coefficients. Existence theorem allows for stochastic white-noise perturbations of the memory, e.g.

$$dx(t) = \left\{ \int_{[-r,0]} x(t+s)\,dW(s) \right\} dW(t) \qquad t > 0$$

   Above sfde is *not* covered by classical results of Protter, Metivier and Pellaumail, Doleans-Dade.

3. Mean Lipschitz, smooth and sublinear dependence of the trajectory random field.

Lecture II. Markov Behavior and the Generator.

1. Markov (Feller) property holds for the trajectory random field. Time homogeneity.
2. Construction of the semigroup. Semigroup is not strongly continuous for positive delay. Domain of strong continuity does not contain tame (or cylinder) functions with evaluations away from 0, but contains "quasitame" functions. These are weakly dense in the underlying space of continuous functions and generate the Borel σ-algebra of the state space.
3. Derivation of a formula for the weak infinitesimal generator of the semigroup for sufficiently regular functions, and for a large class of quasitame functions.

Lecture III. Regularity. Classification of SFDE's.

1. Pathwise regularity of the trajectory random field in the time variable. α-Hölder continuity.
2. Almost sure (pathwise) dependence on the initial state. Non-existence of the stochastic flow for the singular sdde $dx(t) = x(t-r)\,dW(t)$. Breakdown of linearity and local boundedness. Classification of sfde's into regular and singular types.
3. Results on sufficient conditions for regularity of linear systems driven by white noise or semimartingales.
4. Sussman-Doss type nonlinear sfde's. Existence and compactness of semiflow.
5. Path regularity of general non-linear sfde's with "smooth memory".

Lecture IV. Ergodic Theory of Linear SFDE's.

1. Existence of stochastic semiflows for certain classes of linear sfde's with smooth memory terms. The cocycle and its perfection.
2. Compactness of the semiflow in the finite memory case.
3. Ruelle-Oseledec multiplicative ergodic theorem in Hilbert space. Existence of a discrete Lyapunov spectrum. The Stable Manifold Theorem (viz. *random saddles*) for hyperbolic linear sfde's driven by white noise. The case of helix noise.

Lecture V. Stability. Examples and Case Studies.

1. Estimates on the maximal exponential growth rate for the singular noisy feedback loop: $dx(t) = \sigma x(t-r)\,dW(t)$. Stability and instability for small σ (or large r) using a Lyapunov functional argument. Comparison with the non-delay case for large σ.
2. Derivation of estimates on the top Lyapunov exponent λ_1 for various examples of one-dimensional regular sfde's driven by white noise or a martingale with stationary ergodic increments.
3. Lyapunov spectrum for sdde $dx(t) = x((t-1)-)\,dN(t)$ driven by a Poisson process N. Characterization of the Lyapunov spectrum.

Lecture VI. Miscellany

1. Malliavin Calculus of SFDE's. Regularity of the solution $x(t,\omega)$ in ω. Malliavin smoothness and existence of smooth densities. Classical solution of a degenerate parabolic pde as an application.
2. Small delays. Applications to sode's. A proof of the classical existence theorem for solutions to sode's.
3. Affine systems of sfde's. Lyapunov spectrum. The hyperbolic splitting. Existence of stationary solutions in the hyperbolic case. Application to simple population model.
4. Random delays. Induced measure-valued process. Random families of Markov fields and random generators.
5. Infinite memory and stationary solutions.

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I. EXISTENCE

Geilo, Norway

Monday, July 29, 1996

14:00-14:50

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I. EXISTENCE

1. Examples

Example 1. (*Noisy Feedbacks*)

Box N: Input $= y(t)$, output $= x(t)$ at time $t > 0$ related by

$$x(t) = x(0) + \int_0^t y(u)\, dZ(u) \tag{1}$$

where $Z(u)$ is a semimartingale noise.

Box D: Delays signal $x(t)$ by $r\,(> 0)$ units of time. A proportion σ $(0 \le \sigma \le 1)$ is transmitted through D and the rest $(1 - \sigma)$ is used for other purposes.

Therefore

$$y(t) = \sigma x(t - r)$$

Take $\dot{Z}(u) :=$ white noise $= \dot{W}(u)$

Then substituting in (1) gives the Itô integral equation

$$x(t) = x(0) + \sigma \int_0^t x(u-r) dW(u)$$

or the stochastic differenial delay equation (sdde):

$$dx(t) = \sigma x(t-r) dW(t), \qquad t > 0 \qquad (I)$$

To solve (I), need an *initial process* $\theta(t)$, $-r \leq t \leq 0$:

$$x(t) = \theta(t) \quad \text{a.s.}, \qquad -r \leq t \leq 0$$

$\mathbf{r} = \mathbf{0}$: (I) becomes a linear stochastic ode and has closed form solution

$$x(t) = x(0) e^{\sigma W(t) - \frac{\sigma^2 t}{2}}, \qquad t \geq 0.$$

r>0: Solve (I) by successive Itô integrations over steps of length r:

$$\begin{aligned}
x(t) &= \theta(0) + \sigma \int_0^t \theta(u-r)\, dW(u), \quad 0 \leq t \leq r \\
x(t) &= x(r) + \sigma \int_r^t [\theta(0) + \sigma \int_0^{(v-r)} \theta(u-r)\, dW(u)]\, dW(v),\ r < t \leq 2r, \\
\cdots &= \cdots \qquad\qquad 2r < t \leq 3r,
\end{aligned}$$

No closed form solution is known (even in deterministic case).

Curious Fact!

In the sdde (I) the Itô differential dW may be replaced by the Stratonovich differential $\circ dW$ *without changing the solution* x. Let x be the solution of (I) under an Itô differential dW. Then using finite partitions $\{u_k\}$ of the interval $[0,t]$:

$$\int_0^t x(u-r) \circ dW(t) = \lim \sum_k \frac{1}{2} [x(u_k - r) + x(u_{k+1} - r)][W(u_{k+1}) - W(u_k)]$$

where the limit in probability is taken as the mesh of the partition $\{u_k\}$ goes to zero. Compare the Stratonovich and Itô integrals using the corresponding partial sums:

$$\begin{aligned}
\lim E\Bigg(\sum_k \frac{1}{2}[x(u_k - r) + x(u_{k+1} - r)][W(u_{k+1}) - W(u_k)] \\
- \sum_k [x(u_k - r)][W(u_{k+1}) - W(u_k)]\Bigg)^2 \\
= \lim E\Bigg(\sum_k \frac{1}{2}[x(u_{k+1} - r) - x(u_k - r)][W(u_{k+1}) - W(u_k)]\Bigg)^2 \\
= \lim \sum_k \frac{1}{4} E[x(u_{k+1} - r) - x(u_k - r)]^2\, E[W(u_{k+1}) - W(u_k)]^2 \\
= \lim \sum_k \frac{1}{4} E[x(u_{k+1} - r) - x(u_k - r)]^2\, (u_{k+1} - u_k) \\
= 0
\end{aligned}$$

because W has independent increments, x is adapted to the Brownian filtration, $u \mapsto x(u) \in L^2(\Omega, \mathbf{R})$ is continuous, and the delay r is positive. Alternatively

$$\int_0^t x(u - r) \circ dW(u) = \int_0^t x(u - r)\, dW(u) + \frac{1}{2} < x(\cdot - r, W > (t)$$

and $< x(\cdot - r, W > (t) = 0$ for all $t > 0$.

Remark.

When $r > 0$, the solution process $\{x(t) : t \geq -r\}$ of (I) is a martingale but is *non-Markov*.

Example 2. (*Simple Population Growth*)

Consider a large population $x(t)$ at time t evolving with a constant birth rate $\beta > 0$ and a constant death rate α per capita. Assume immediate removal of the dead from the population. Let $r > 0$ (fixed, non-random$= 9$, e.g.) be the development period of each individual and assume there is migration whose overall rate is distributed like white noise $\sigma\dot{W}$ (mean zero and variance $\sigma > 0$), where W is one-dimensional standard Brownian motion. The change in population $\Delta x(t)$ over a small time interval $(t, t+\Delta t)$ is

$$\Delta x(t) = -\alpha x(t)\Delta t + \beta x(t-r)\Delta t + \sigma\dot{W}\Delta t$$

Letting $\Delta t \to 0$ and using Itô stochastic differentials,

$$dx(t) = \{-\alpha x(t) + \beta x(t-r)\}\, dt + \sigma dW(t), \quad t > 0. \tag{II}$$

Associate with the above affine sdde the initial condition $(v, \eta) \in \mathbf{R} \times L^2([-r, 0], \mathbf{R})$

$$x(0) = v, \quad x(s) = \eta(s), \quad -r \le s < 0.$$

Denote by $M_2 = \mathbf{R} \times L^2([-r, 0], \mathbf{R})$ the Delfour-Mitter Hilbert space of all pairs (v, η), $v \in \mathbf{R}$, $\eta \in L^2([-r, 0], \mathbf{R})$ with norm

$$\|(v, \eta)\|_{M_2} = \left(|v|^2 + \int_{-r}^{0} |\eta(s)|^2\, ds\right)^{1/2}.$$

Let $W : \mathbf{R}^+ \times \Omega \to \mathbf{R}$ be defined on the canonical filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbf{R}^+}, P)$ where

$$\Omega = C(\mathbf{R}^+, \mathbf{R}),\ \mathcal{F} = \text{ Borel } \Omega,\ \mathcal{F}_t = \sigma\{\rho_u : u \le t\}$$

$\rho_u : \Omega \to \mathbf{R}, u \in \mathbf{R}^+$, are evaluation maps $\omega \mapsto \omega(u)$, and $P =$ Wiener measure on Ω.

Example 3. (*Logistic Population Growth*)

A single population $x(t)$ at time t evolving logistically with *development (incubation) period* $r > 0$ under Gaussian type noise (e.g. migration on a molecular level):

$$\dot{x}(t) = [\alpha - \beta x(t-r)]\, x(t) + \gamma x(t)\dot{W}(t), \quad t > 0$$

i.e.

$$dx(t) = [\alpha - \beta x(t-r)]\, x(t)\, dt + \gamma x(t) dW(t) \qquad t > 0. \qquad (III)$$

with *initial condition*

$$x(t) = \theta(t) \qquad -r \leq t \leq 0.$$

For positive delay r the above sdde can be solved *implicitly* using forward steps of length r, i.e. for $0 \leq t \leq r$, $x(t)$ satisfies the *linear* sode (without delay)

$$dx(t) = [\alpha - \beta \theta(t-r)]\, x(t)\, dt + \gamma x(t) dW(t) \qquad 0 < t \leq r. \qquad (III')$$

$x(t)$ is a semimartingale and is non-Markov (Scheutzow [S], 1984).

Example 4. (*Heat bath*)

Model proposed by R. Kubo (1966) for physical Brownian motion. A molecule of mass m moving under random gas forces with position $\xi(t)$ and velocity $v(t)$ at time t; cf classical work by Einstein and Ornestein and Uhlenbeck. Kubo proposed the following modification of the Ornstein-Uhenbeck process

$$\left.\begin{aligned} d\xi(t) &= v(t)\, dt \\ m dv(t) &= -m[\int_{t_0}^{t} \beta(t-t') v(t')\, dt']\, dt + \gamma(\xi(t), v(t))\, dW(t), \ t > t_0. \end{aligned}\right\} \qquad (IV)$$

m =mass of molecule. No external forces.

$\beta =$ viscosity coefficient function with compact support.

γ a function $\mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}$ representing the random gas forces on the molecule.

$\xi(t) =$ position of molecule $\in \mathbf{R}^3$.

$v(t) =$ velocity of molecule $\in \mathbf{R}^3$.

$W = 3-$ dimensional Brownian motion.

([Mo], Pitman Books, RN # 99, 1984, pp. 223-226).

Further Examples

Delay equation with Poisson noise:

$$\left.\begin{aligned} dx(t) &= x((t-r)-)\,dN(t) \quad t>0 \\ x_0 &= \eta \in D([-r,0],\mathbf{R}) \end{aligned}\right\} \tag{V}$$

$N :=$ Poisson process with iid interarrival times ([S], Hab. 1988). $D([-r,0],\mathbf{R}) =$ space of all cadlag paths $[-r,0] \to \mathbf{R}$, with sup norm.

Simple model of dye circulation in the blood (or pollution) (cf. Bailey and Williams [B-W], JMAA, 1966, Lenhart and Travis ([L-T], PAMS, 1986).

$$\left.\begin{aligned} dx(t) &= \{\nu x(t) + \mu x(t-r))\}\,dt + \sigma x(t)\,dW(t) \quad t>0 \\ (x(0),x_0) &= (v,\eta) \in M_2 = \mathbf{R} \times L^2([-r,0],\mathbf{R}), \end{aligned}\right\} \tag{VI}$$

([Mo], Survey, 1992; [M-S], II, 1995.)

In above model:

$x(t) :=$ dye concentration (gm/cc)

$r =$ time taken by blood to traverse side tube (vessel)

Flow rate (cc/sec) is Gaussian with variance σ.

A fixed proportion of blood in main vessel is pumped into side vessel(s). Model will be analysed in Lecture V (Theorem V.5).

$$\left.\begin{aligned} dx(t) &= \{\nu x(t) + \mu x(t-r))\}\,dt + \{\int_{-r}^{0} x(t+s)\sigma(s)\,ds\}\,dW(t), \\ (x(0), x_0) &= (v, \eta) \in M_2 = \mathbf{R} \times L^2([-r,0], \mathbf{R}),\ t > 0. \end{aligned}\right\} \qquad (VII)$$

([Mo], Survey, 1992; [M-S], II, 1995.)

Linear d-dimensional systems driven by m-dimensional Brownian motion $W := (W_1, \cdots, W_m)$ with constant coefficients.

$$\left.\begin{aligned} dx(t) &= H(x(t-d_1), \cdots, x(t-d_N), x(t), x_t)dt \\ &\qquad + \sum_{i=1}^{m} g_i x(t)\,dW_i(t), \quad t > 0 \\ (x(0), x_0) &= (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r,0], \mathbf{R}^d) \end{aligned}\right\} \qquad (VIII)$$

$H := (\mathbf{R}^d)^N \times M_2 \to \mathbf{R}^d$ linear functional on $(\mathbf{R}^d)^N \times M_2$; g_i $d \times d$-matrices ([Mo], Stochastics, 1990).

Linear systems driven by (helix) semimartingale noise (N, L), and memory driven by a (stationary) measure-valued process ν and a (stationary) process K ([M-S], I, AIHP, 1996):

$$\left.\begin{aligned} dx(t) &= \Big\{\int_{[-r,0]} \nu(t)(ds)\,x(t+s)\Big\}\,dt \\ &\quad + dN(t) \int_{-r}^{0} K(t)(s)\,x(t+s)\,ds + dL(t)\,x(t-), \quad t > 0 \\ (x(0), x_0) &= (v, \eta) \in M_2 = \mathbf{R}^d \times L^2([-r,0], \mathbf{R}^d) \end{aligned}\right\} \qquad (IX)$$

Multidimensional affine systems driven by (helix) noise Q ([M-S], Stochastics, 1990):

$$\left.\begin{aligned} dx(t) &= \Big\{\int_{[-r,0]} \nu(t)(ds)\,x(t+s)\Big\}\,dt + dQ(t), \quad t > 0 \\ (x(0), x_0) &= (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r,0], \mathbf{R}^d) \end{aligned}\right\} \qquad (X)$$

Memory driven by white noise:

$$\left.\begin{array}{l} dx(t) = \left\{ \int_{[-r,0]} x(t+s)\, dW(s) \right\} dW(t) \qquad t > 0 \\ x(0) = v \in \mathbf{R}, \quad x(s) = \eta(s), \quad -r < s < 0, \quad r \geq 0 \end{array}\right\} \tag{XI}$$

([Mo], Survey, 1992).

Formulation

Slice each solution path x over the interval $[t-r, t]$ to get *segment* x_t as a process on $[-r, 0]$:

$$x_t(s) := x(t+s) \qquad \text{a.s., } t \geq 0, s \in J := [-r, 0].$$

Therefore sdde's (I), (II), (III) and (XI) become

$$\left.\begin{array}{r} dx(t) = \sigma x_t(-r) dW(t), \quad t > 0 \\ \\ x_0 = \theta \in C([-r, 0], \mathbf{R}) \end{array}\right\} \qquad (I)$$

$$\left.\begin{array}{r} dx(t) = \{-\alpha x(t) + \beta x_t(-r)\}\, dt + \sigma dW(t), \quad t > 0 \\ \\ (x(0), x_0) = (v, \eta) \in \mathbf{R} \times L^2([-r, 0], \mathbf{R}) \end{array}\right\} \qquad (II)$$

$$\left.\begin{array}{l} dx(t) = [\alpha - \beta x_t(-r)]x_t(0)\,dt + \gamma x_t(0)\,dW(t) \\ \\ x_0 = \theta \in C([-r,0],\mathbf{R}) \end{array}\right\} \qquad (III)$$

$$\left.\begin{array}{r} dx(t) = \left\{\int_{[-r,0]} x_t(s)\,dW(s)\right\} dW(t) \quad t > 0 \\ (x(0), x_0) = (v,\eta) \in \mathbf{R} \times L^2([-r,0],\mathbf{R}), \quad r \geq 0 \end{array}\right\} \qquad (XI)$$

Think of R.H.S.'s of the above equations as functionals of x_t (and $x(t)$) and generalize to *stochastic functional differential equation* (sfde)

$$\left.\begin{array}{l} dx(t) = h(t,x_t)dt + g(t,x_t)dW(t) \quad t > 0 \\ \\ x_0 = \theta \end{array}\right\} \qquad (XII)$$

on filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ satisfying the usual conditions:

$(\mathcal{F}_t)_{t\geq 0}$ right-continuous and each $\mathcal{F}_t$ contains all P-null sets in $\mathcal{F}$.

$C := C([-r,0],\mathbf{R}^d)$ Banach space, sup norm.

$W(t) = m$–dimensional Brownian motion.

$L^2(\Omega, C) :=$ Banach space of all $(\mathcal{F}, Borel C)$-measurable L^2 (Bochner sense) maps $\Omega \to C$ with the L^2-norm

$$\|\theta\|_{L^2(\Omega,C)} := \left[\int_\Omega \|\theta(\omega)\|_C^2 \, dP(\omega)\right]^{1/2}$$

Coefficients:

$$h : [0,T] \times L^2(\Omega, C) \to L^2(\Omega, \mathbf{R}^d) \qquad \text{(Drift)}$$

$$g : [0,T] \times L^2(\Omega, C) \to L^2(\Omega, L(\mathbf{R}^m, \mathbf{R}^d) \qquad \text{(Diffusion).}$$

Initial data:

$$\theta \in L^2(\Omega, C, \mathcal{F}_0).$$

Solution:

$x : [-r, T] \times \Omega \to \mathbf{R}^d$ measurable and sample-continuous, $x|[0,T]$ $(\mathcal{F}_t)_{0 \le t \le T}$-adapted and $x(s)$ is $\mathcal{F}_0$-measurable for all $s \in [-r, 0]$.

Exercise: $[0,T] \ni t \mapsto x_t \in C([-r,0], \mathbf{R}^d)$ is $(\mathcal{F}_t)_{0 \le t \le T}$-adapted.

(*Hint:* $Borel\, C$ is generated by all evaluations.)

Hypotheses (E_1)**.**

(i) h, g are jointly continuous and uniformly Lipschitz in the second variable with respect to the first:

$$\|h(t,\psi_1) - h(t,\psi_2)\|_{L^2(\Omega,\mathbf{R}^d)} \le L\|\psi_1 - \psi_2\|_{L^2(\Omega,C)}$$

for all $t \in [0,T]$ and $\psi_1, \psi_2 \in L^2(\Omega, C)$. Similarly for the diffusion coefficent g.

(ii) For each $(\mathcal{F}_t)_{0\le t\le T}$-adapted process $y : [0,T] \to L^2(\Omega, C)$,

the processes $h(\cdot, y(\cdot)), g(\cdot, y(\cdot))$ are also $(\mathcal{F}_t)_{0\le t\le T}$- adapted.

Theorem I.1. ([Mo], 1984) (Existence and Uniqueness).

Suppose h and g satisfy Hypotheses (E_1). Let $\theta \in L^2(\Omega, C; \mathcal{F}_0)$.

Then the sfde (XII) has a unique solution ${}^\theta x : [-r,\infty) \times \Omega \to \mathbf{R}^d$ starting off at $\theta \in L^2(\Omega, C; \mathcal{F}_0)$ with $t \longmapsto {}^\theta x_t$ continuous and ${}^\theta x \in L^2(\Omega, C([-r,T]\mathbf{R}^d))$ for all $T > 0$. For a given θ, uniqueness holds up to equivalence among all $(\mathcal{F}_t)_{0\le t\le T}$-adapted processes in $L^2(\Omega, C([-r,T], \mathbf{R}^d))$.

Proof.

[Mo], Pitman Books, 1984, Theorem 2.1, pp. 36-39. □

Theorem I.1 covers equations (I), (II), (IV), (VI), (VII), (VIII), (XI) and a large class of sfde's driven by white noise. Note that (XI) *does not satisfy the hypotheses underlying the classical results* of Doleans-Dade [Dol], 1976, Metivier and Pellaumail [Met-P], 1980, Protter, Ann. Prob. 1987, Lipster and Shiryayev [Lip-Sh], [Met], 1982. This is because the coefficient

$$\eta \to \int_{-r}^{0} \eta(s)\, dW(s)$$

on the RHS of (XI) *does not admit almost surely Lipschitz (or even linear) versions* $C \to \mathbf{R}$! This will be shown later.

When the coeffcients h, g factor through functionals

$$H : [0,T] \times C \to \mathbf{R}^d, \qquad G : [0,T] \times C \to \mathbf{R}^{d \times m}$$

we can impose the following local Lipschitz and global linear growth conditions on the sfde

$$\left.\begin{aligned} dx(t) &= H(t, x_t)\, dt + G(t, x_t)\, dW(t) \quad t > 0 \\ x_0 &= \theta \end{aligned}\right\} \qquad (XIII)$$

with W m-dimensional Brownian motion:

Hypotheses (E_2)

(i) H, G are Lipschitz on bounded sets in C: For each integer $n \geq 1$ there exists $L_n > 0$ such that

$$|H(t,\eta_1) - H(t,\eta_2)| \leq L_n \|\eta_1 - \eta_2\|_C$$

for all $t \in [0,T]$ and $\eta_1, \eta_2 \in C$ with $\|\eta_1\|_C \leq n$, $\|\eta_2\|_C \leq n$. Similarly for the diffusion coefficent G.

(ii) There is a constant $K > 0$ such that

$$|H(t,\eta)| + \|G(t,\eta)\| \leq K(1 + \|\eta\|_C)$$

for all $t \in [0,T]$ and $\eta \in C$.

Note that the adaptability condition is not needed (explicitly) because H, G are deterministic and because the sample-continuity and adaptability of x imply that the segment $[0,T] \ni t \mapsto x_t \in C$ is also adapted.

Exercise: Formulate the heat-bath model (IV) as a sfde of the form (XIII).(β has compact support in $\mathbf{R}^+$.)

Theorem I.2. ([Mo], 1984) (Existence and Uniqueness).

Suppose H and G satisfy Hypotheses (E_2) and let $\theta \in L^2(\Omega, C; \mathcal{F}_0)$.

Then the sfde (XIII) has a unique $(\mathcal{F}_t)_{0\le t\le T}$-adapted solution ${}^\theta x : [-r,T]\times\Omega \to \mathbf{R}^d$ starting off at $\theta \in L^2(\Omega, C; \mathcal{F}_0)$ with $t \longmapsto {}^\theta x_t$ continuous and ${}^\theta x \in L^2(\Omega, C([-r,T],\mathbf{R}^d))$ for all $T > 0$. For a given θ, uniqueness holds up to equivalence among all $(\mathcal{F}_t)_{0\le t\le T}$-adapted processes in $L^2(\Omega, C([-r,T],\mathbf{R}^d))$.

Furthermore if $\theta \in L^{2k}(\Omega, C; \mathcal{F}_0)$, then ${}^\theta x_t \in L^{2k}(\Omega, C; \mathcal{F}_t)$ and

$$E\|{}^\theta x_t\|_C^{2k} \le C_k[1 + \|\theta\|_{L^{2k}(\Omega,C)}^{2k}]$$

for all $t \in [0,T]$ and some positive constants C_k.

Proofs of Theorems I.1, I.2.(Outline)

[Mo], pp. 150-152. Generalize sode proofs in Gihman and Skorohod ([G-S], 1973) or Friedman ([Fr], 1975):

(1) Truncate coefficients outside bounded sets in C. Reduce to globally Lipschitz case.

(2) Successive approx. in globally Lipschitz situation.

(3) Use local uniqueness ([Mo], Theorem 4.2, p. 151) to "patch up" solutions of the truncated sfde's.

For (2) consider globally Lipschitz case and $h \equiv 0$.

We look for solutions of (XII) by successive approximation in $L^2(\Omega, C([-r,a], \mathbf{R}^d))$. Let $J := [-r, 0]$.

Suppose $\theta \in L^2(\Omega, C(J, \mathbf{R}^d))$ is $\mathcal{F}_0$-measurable. Note that this is equivalent to saying that $\theta(\cdot)(s)$ is $\mathcal{F}_0$-measurable for all $s \in J$, because θ has a.a. sample paths continuous.

We prove by induction that there is a sequence of processes ${}^k x : [-r, a] \times \Omega \to \mathbf{R}^d$, $k = 1, 2, \cdots$ having the

Properties $P(k)$*:*

(i) ${}^{k}x \in L^2(\Omega, C([-r,a], \mathbf{R}^d))$ and is adapted to $(\mathcal{F}_t)_{t\in[0,a]}$.

(ii) For each $t \in [0,a]$, ${}^{k}x_t \in L^2(\Omega, C(J, \mathbf{R}^d))$ and is $\mathcal{F}_t$-measur-able.

(iii)

$$\left.\begin{aligned} \|{}^{k+1}x - {}^{k}x\|_{L^2(\Omega,C)} &\leq (ML^2)^{k-1}\frac{a^{k-1}}{(k-1)!}\|{}^{2}x - {}^{1}x\|_{L^2(\Omega,C)} \\ \|{}^{k+1}x_t - {}^{k}x_t\|_{L^2(\Omega,C)} &\leq (ML^2)^{k-1}\frac{t^{k-1}}{(k-1)!}\|{}^{2}x - {}^{1}x\|_{L^2(\Omega,C)} \end{aligned}\right\} \tag{1}$$

where M is a "martingale" constant and L is the Lipschitz constant of g.

Take ${}^{1}x : [-r,a] \times \Omega \to \mathbf{R}^d$ to be

$$^{1}x(t,\omega) = \begin{cases} \theta(\omega)(0) & t \in [0,a] \\ \theta(\omega)(t) & t \in J \end{cases}$$

a.s., and

$$^{k+1}x(t,\omega) = \begin{cases} \theta(\omega)(0) + (\omega)\displaystyle\int_0^t g(u, {}^{k}x_u)dW(\cdot)(u) & t \in [0,a] \\ \theta(\omega)(t) \qquad t \in J \end{cases} \tag{2}$$

a.s.

Since $\theta \in L^2(\Omega, C(J, \mathbf{R}^d))$ and is $\mathcal{F}_0$-measurable, then ${}^{1}x \in L^2(\Omega, C([-r,a], \mathbf{R}^d))$ and is trivially adapted to $(\mathcal{F}_t)_{t\in[0,a]}$. Hence ${}^{1}x_t \in L^2(\Omega, C(J, \mathbf{R}^d))$ and is $\mathcal{F}_t$-measurable for all $t \in [0,a]$. $P(1)$ (iii) holds trivially.

Now suppose $P(k)$ is satisfied for some $k > 1$. Then by Hypothesis $(E_1)(i), (ii)$ and the continuity of the slicing map (*stochastic memory*), it follows from $P(k)(ii)$ that the process

$$[0,a] \ni u \longmapsto g(u, {}^k x_u) \in L^2(\Omega, L(\mathbf{R}^m, \mathbf{R}^d))$$

is continuous and adapted to $(\mathcal{F}_t)_{t\in[0,a]}$. $P(k+1)(i)$ and $P(k+1)(ii)$ follow from the continuity and adaptability of the stochastic integral. Check $P(k+1)(iii)$, by using Doob's inequality.

For each $k > 1$, write

$${}^k x = {}^1 x + \sum_{i=1}^{k-1} ({}^{i+1}x - {}^i x).$$

Now $L^2_A(\Omega, C([-r,a], \mathbf{R}^d))$ is closed in $L^2(\Omega, C([-r,a], \mathbf{R}^d))$; so the series

$$\sum_{i=1}^{\infty} ({}^{i+1}x - {}^i x)$$

converges in $L^2_A(\Omega, C([-r,a], \mathbf{R}^d))$ because of (1) and the convergence of

$$\sum_{i=1}^{\infty} \left[(ML^2)^{i-1} \frac{a^{i-1}}{(i-1)!} \right]^{1/2}.$$

Hence $\{{}^k x\}_{k=1}^{\infty}$ converges to some $x \in L^2_A(\Omega, C([-r,a], \mathbf{R}^d))$.

Clearly $x|J = \theta$ and is $\mathcal{F}_0$-measurable, so applying Doob's inequality to the Itô integral of the difference

$$u \longmapsto g(u, {}^k x_u) - g(u, x_u)$$

gives

$$\begin{aligned} E\bigg(\sup_{t\in[0,a]} \bigg| \int_0^t g(u, {}^k x_u)\, dW(\cdot)(u) - \int_0^t g(u, x_u)\, dW(\cdot)(u) \bigg|^2\bigg) \\ &< ML^2 a \|{}^k x - x\|^2_{L^2(\Omega, C)} \\ &\longrightarrow 0 \text{ as } k \to \infty. \end{aligned}$$

Thus viewing the right-hand side of (2) as a process in $L^2(\Omega, C\,([-r,a], \mathbf{R}^d))$ and letting $k \to \infty$, it follows from the above that x must satisfy the sfde (XII) a.s. for all $t \in [-r, a]$.

For uniqueness, let $\tilde{x} \in L^2_A(\Omega, ([-r,a], \mathbf{R}^d))$ be also a solution of (XII) with initial process θ. Then by the Lipschitz condition:

$$\|x_t - \tilde{x}_t\|^2_{L^2(\Omega,C)} < ML^2 \int_0^t \|x_u - \tilde{x}_u\|^2_{L^2(\Omega,C)}\, du$$

for all $t \in [0,a]$. Therefore we must have $x_t - \tilde{x}_t = 0$ for all $t \in [0,a]$; so $x = \tilde{x}$ in $L^2(\Omega, C([-r,a], \mathbf{R}^d))$ a.s. □

Remarks and Generalizations.

(i) In Theorem I.2 replace the process $(t, W(t))$ by a (square integrable) semimartingale $Z(t)$ satisfying appropriate conditions.([Mo], 1984, Chapter II).

(ii) Results on existence of solutions of sfde's driven by white noise were first obtained by Itô and Nisio ([I-N], J. Math. Kyoto University, 1968) and then Kushner (JDE, 197).

(iii) Extensions to sfde's with *infinite* memory. Fading memory case: work by Mizel and Trützer [M-T],JIE, 1984, Marcus and Mizel [M-M], Stochastics, 1988; general infinite memory: Itô and Nisio [I-N], J. Math. Kyoto University, 1968.

(iii) Pathwise local uniqueness holds for sfde's of type (XIII) under a global Lipschitz condition: If coeffcients of two sfde's agree on an open set in C, then the corresponding trajectories leave the open set at the same time and agree almost surely up to the time they leave the open set ([Mo], Pitman Books, 1984, Theorem 4.2, pp. 150-151.)

(iv) Replace the state space C by the Delfour-Mitter Hilbert space $M_2 := \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d)$ with the Hilbert norm

$$\|(v, \eta)\|_{M_2} = \left(|v|^2 + \int_{-r}^{0} |\eta(s)|^2 \, ds \right)^{1/2}$$

for $(v, \eta) \in M_2$ (T. Ahmed, S. Elsanousi and S. Mohammed, 1983).

(v) Have Lipschitz and smooth dependence of ${}^{\theta}x_t$ on the initial process $\theta \in L^2(\Omega, C)$ ([Mo], 1984, Theorems 3.1, 3.2, pp. 41-45).

II. MARKOV BEHAVIOR
AND THE WEAK GENERATOR

Geilo, Norway

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14:00-14:50

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II. MARKOV BEHAVIOR AND THE GENERATOR

Consider the sfde

$$\left.\begin{aligned} dx(t) &= H(t, x_t)\, dt + G(t, x_t)\, dW(t), \qquad t > 0 \\ x_0 &= \eta \in C := C([-r, 0], \mathbf{R}^d) \end{aligned}\right\} \qquad (XIII)$$

with coefficients $H : [0,T]\times C \to \mathbf{R}^d$, $G : [0,T]\times C \to \mathbf{R}^{d\times m}$, m-dimensional Brownian motion W and trajectory field $\{{}^\eta x_t : t \geq 0, \eta \in C\}$.

1. Questions

(i) For the sfde (XIII) does the trajectory field x_t give a diffusion in C (or M_2)?

(ii) How does the trajectory x_t transform under smooth non-linear functionals $\phi : C \to \mathbf{R}$?

(iii) What "diffusions" on C (or M_2) correspond to sfde's on $\mathbf{R}^d$?

We will only answer the first two questions. More details in [Mo], Pitman Books, 1984, Chapter III, pp. 46-112. Third question is OPEN.

Difficulties

(i) Although the current state $x(t)$ is a semimartingale, the trajectory x_t does *not* seem to possess any martingale properties when viewed as C-(or M_2)-valued process: e.g. for Brownian motion W $\big(H \equiv 0, G \equiv 1\big)$:

$$[E(W_t|\mathcal{F}_{t_1})](s) = W(t_1) = W_{t_1}(0), \qquad s \in [-r, 0]$$

whenever $t_1 \leq t - r$.

(ii) Lack of strong continuity leads to the use of weak limits in C which tend to live outside C.

(iii) We will show that x_t is a Markov process in C. However almost all tame functions lie *outside* the domain of the (weak) generator.

(iv) Lack of an Itô formula makes the computation of the generator hard.

Hypotheses (M)

(i) $\mathcal{F}_t :=$ completion of $\sigma\{W(u) : 0 \leq u \leq t\}, \quad t \geq 0$.

(ii) H, G are jointly continuous and globally Lipschitz in second variable uniformly wrt the first:

$$|H(t,\eta_1) - H(t,\eta_2)| + \|G(t,\eta_1) - G(t,\eta_2)\| \leq L\|\eta_1 - \eta_2\|_C$$

for all $t \in [0,T]$ and $\eta_1, \eta_2 \in C$.

2. The Markov Property

${}^{\eta}x^{t_1} :=$ solution starting off at $\theta \in L^2(\Omega, C; \mathcal{F}_{t_1})$ at $t = t_1$ for the sfde:

$${}^{\eta}x^{t_1}(t) = \begin{cases} \eta(0) + \int_{t_1}^{t} H(u, x_u^{t_1})\, du + \int_{t_1}^{t} G(u, x_u^{t_1})\, dW(u), & t > t_1 \\ \eta(t - t_1), & t_1 - r \leq t \leq t_1. \end{cases}$$

This gives a two-parameter family of mappings

$$T^{t_1}_{t_2} : L^2(\Omega, C; \mathcal{F}_{t_1}) \to L^2(\Omega, C; \mathcal{F}_{t_2}),\ t_1 \le t_2,$$

$$T^{t_1}_{t_2}(\theta) := {}^{\theta}x^{t_1}_{t_2}, \qquad \theta \in L^2(\Omega, C; \mathcal{F}_{t_1}). \tag{1}$$

Uniqueness of solutions gives the *two-parameter* semigroup property:

$$T^{t_1}_{t_2} \circ T^0_{t_1} = T^0_{t_2}, \quad t_1 \le t_2. \tag{2}$$

([Mo], Pitman Books, 1984, Theorem II (2.2), p. 40.)

Theorem II.1 (Markov Property)([Mo], 1984).

In (XIII) suppose Hypotheses (M) hold. Then the trajectory field $\{{}^{\eta}x_t : t \ge 0, \eta \in C\}$ is a Feller process on C with transition probabilities

$$p(t_1, \eta, t_2, B) := P\big({}^{\eta}x^{t_1}_{t_2} \in B\big) \quad t_1 \le t_2, \quad B \in \text{ Borel } C, \quad \eta \in C.$$

i.e.

$$P\big(x_{t_2} \in B \big| \mathcal{F}_{t_1}\big) = p(t_1, x_{t_1}(\cdot), t_2, B) = P\big(x_{t_2} \in B \big| x_{t_1}\big) \text{ a.s.}$$

Further, if H and G do not depend on t, then the trajectory is time-homogeneous:

$$p(t_1, \eta, t_2, \cdot) = p(0, \eta, t_2 - t_1, \cdot), \quad 0 \le t_1 \le t_2, \quad \eta \in C.$$

Proof.

[Mo], 1984, Theorem III.1.1, pp. 51-58. [Mo], 1984, Theorem III.2.1, pp. 64-65. □

3. The Semigroup

In the autonomous sfde

$$\left.\begin{aligned} dx(t) &= H(x_t)\,dt + G(x_t)\,dW(t) \quad t>0 \\ x_0 &= \eta \in C \end{aligned}\right\} \qquad (XIV)$$

suppose the coefficients $H : C \to \mathbf{R}^d$, $\quad G : C \to \mathbf{R}^{d\times m}$ are *globally bounded* and globally Lipschitz.

$C_b :=$ Banach space of all bounded uniformly continuous functions $\phi : C \to \mathbf{R}$, with the sup norm

$$\|\phi\|_{C_b} := \sup_{\eta\in C} |\phi(\eta)|, \qquad \phi \in C_b.$$

Define the operators $P_t : C_b \hookrightarrow C_b, t \geq 0$, on C_b by

$$P_t(\phi)(\eta) := E\phi\big({}^\eta x_t\big) \qquad t\geq 0,\ \eta \in C.$$

A family $\phi_t, t > 0$, *converges weakly* to $\phi \in C_b$ as $t \to 0+$ if $\lim\limits_{t\to 0+} < \phi_t, \mu > = < \phi, \mu >$ for all finite regular Borel measures μ on C. Write $\phi := w - \lim\limits_{t\to 0+} \phi_t$. This is equivalent to

$$\begin{cases} \phi_t(\eta) \to \phi(\eta) \text{ as } t \to 0+, \text{ for all } \eta \in C \\ \\ \{\|\phi_t\|_{C_b} : t \geq 0\} \text{ is bounded }. \end{cases}$$

(Dynkin, [Dy], Vol. 1, p. 50). Proof uses uniform boundedness principle and dominated convergence theorem.

Theorem II.2([Mo], Pitman Books, 1984)

(i) $\{P_t\}_{t\geq 0}$ *is a one-parameter contraction semigroup on* C_b.

(ii) $\{P_t\}_{t\geq 0}$ *is weakly continuous at* $t = 0$*:*

$$\begin{cases} P_t(\phi)(\eta) \to \phi(\eta) \text{ as } t \to 0+ \\ \\ \{|P_t(\phi)(\eta)| : t \geq 0, \eta \in C\} \text{is bounded by } \|\phi\|_{C_b}. \end{cases}$$

(iii) If $r > 0$, $\{P_t\}_{t\geq 0}$ *is never strongly continuous on* C_b *under the sup norm.*

Proof.

(i) One parameter semigroup property

$$P_{t_2} \circ P_{t_1} = P_{t_1+t_2}, \quad t_1, t_2 \geq 0$$

follows from the continuation property (2) and time-homogeneity of the Feller process x_t (Theorem II.1).

(ii) Definition of P_t, continuity and boundedness of ϕ and sample-continuity of trajectory ${}^\eta x_t$ give weak continuity of $\{P_t(\phi) : t > 0\}$ at $t = 0$ in C_b.

(iii) Lack of strong continuity of semigroup:

Define the canonical shift (static) semigroup

$$S_t : C_b \to C_b,\ t \geq 0,$$

by

$$S_t(\phi)(\eta) := \phi(\tilde{\eta}_t), \quad \phi \in C_b, \quad \eta \in C,$$

where $\tilde{\eta} : [-r, \infty) \to \mathbf{R}^d$ is defined by

$$\tilde{\eta}(t) = \begin{cases} \eta(0) & t \geq 0 \\ \eta(t) & t \in [-r, 0). \end{cases}$$

Then P_t is strongly continuous iff S_t is strongly continuous. P_t and S_t have the same "domain of strong continuity" independently of H, G, and W. This follows from the global boundedness of H and G. ([Mo], Theorem IV.2.1, pp. 72-73). Key relation is

$$\lim_{t\to 0+} E\|{}^\eta x_t - \tilde{\eta}_t\|_C^2 = 0$$

uniformly in $\eta \in C$. But $\{S_t\}$ is strongly continuous on C_b iff C is locally compact iff $r = 0$ (no memory) ! ([Mo], Theorems IV.2.1 and IV.2.2, pp.72-73). Main idea is to pick any $s_0 \in [-r, 0)$ and consider the function $\phi_0 : C \to \mathbf{R}$ defined by

$$\phi_0(\eta) := \begin{cases} \eta(s_0) & \|\eta\|_C \leq 1 \\ \dfrac{\eta(s_0)}{\|\eta\|_C} & \|\eta\|_C > 1 \end{cases}$$

Let C_b^0 be the domain of strong continuity of P_t, viz.

$$C_b^0 := \{\phi \in C_b : P_t(\phi) \to \phi \text{ as } t \to 0+ \text{ in } C_b\}.$$

Then $\phi_0 \in C_b$, but $\phi_0 \notin C_b^0$ because $r > 0$. □

4. The Generator

Define the *weak generator* $A : D(A) \subset C_b \to C_b$ by the weak limit

$$A(\phi)(\eta) := w - \lim_{t \to 0+} \frac{P_t(\phi)(\eta) - \phi(\eta)}{t}$$

where $\phi \in D(A)$ iff the above weak limit exists. Hence $D(A) \subset C_0^b$ (Dynkin [Dy], Vol. 1, Chapter I, pp. 36-43). Also $D(A)$ is weakly dense in C_b and A is weakly closed. Further

$$\frac{d}{dt} P_t(\phi) = A(P_t(\phi)) = P_t(A(\phi)), \quad t > 0$$

for all $\phi \in D(A)$ ([Dy], pp. 36-43).

Next objective is to derive a formula for the weak generator A. *We need to augment C by adjoining a canonical d-dimensional direction. The generator A will be equal to the weak generator of the shift semigroup $\{S_t\}$ plus a second order linear partial differential operator along this new direction.* Computation requires the following lemmas.

Let

$$F_d = \{v\chi_{\{0\}} : v \in \mathbf{R}^d\}$$

$$C \oplus F_d = \{\eta + v\chi_{\{0\}} : \eta \in C, v \in \mathbf{R}^d\}, \quad \|\eta + v\chi_{\{0\}}\| = \|\eta\|_C + |v|$$

Lemma II.1.([Mo], Pitman Books, 1984)

Suppose $\phi : C \to \mathbf{R}$ is C^2 and $\eta \in C$. Then $D\phi(\eta)$ and $D^2\phi(\eta)$ have unique weakly continuous linear and bilinear extensions

$$\overline{D\phi(\eta)} : C \oplus F_d \to \mathbf{R}, \quad \overline{D^2\phi(\eta)} : (C \oplus F_d) \times (C \oplus F_d) \to \mathbf{R}$$

respectively.

Proof.

First reduce to the one-dimensional case $d = 1$ by using coordinates.

Let $\alpha \in C^* = [C([-r, 0], \mathbf{R})]^*$. We will show that there is a weakly continuous linear extension $\overline{\alpha} : C \oplus F_1 \to \mathbf{R}$ of α; viz. If $\{\xi^k\}$ is a bounded sequence in C such that $\xi^k(s) \to \xi(s)$ as $k \to \infty$ for all $s \in [-r, 0]$, where $\xi \in C \oplus F_1$, then $\alpha(\xi^k) \to \overline{\alpha}(\xi)$ as $k \to \infty$. By the Riesz representation theorem there is a unique finite regular Borel measure μ on $[-r, 0]$ such that

$$\alpha(\eta) = \int_{-r}^{0} \eta(s)\, d\mu(s)$$

for all $\eta \in C$. Define $\overline{\alpha} \in [C \oplus F_1]^*$ by

$$\overline{\alpha}(\eta + v\chi_{\{0\}}) = \alpha(\eta) + v\mu(\{0\}), \quad \eta \in C, \quad v \in \mathbf{R}.$$

Easy to check that $\overline{\alpha}$ is weakly continuous. (*Exercise:* Use Lebesgue dominated convergence theorem.)

Weak extension $\overline{\alpha}$ is unique because each function $v\chi_{\{0\}}$ can be approximated weakly by a sequence of continuous functions $\{\xi_0^k\}$:

$$\xi_0^k(s) := \begin{cases} (ks+1)v, & -\frac{1}{k} \le s \le 0 \\ 0 & -r \le s < -\frac{1}{k}. \end{cases}$$

Put $\alpha = D\phi(\eta)$ to get first assertion of lemma.

To construct a weakly continuous bilinear extension $\overline{\beta} : (C \oplus F_1) \times (C \oplus F_1) \to \mathbf{R}$ for any continuous bilinear form
$\beta : C \times C \to \mathbf{R}$, use classical theory of vector measures (Dunford and Schwartz, [D-S], Vol. I, Section 6.3). Think of β as a continuos *linear* map $C \to C^*$. Since C^* is weakly complete ([D-S], I.13.22, p. 341), then β is a weakly compact linear operator ([D-S], Theorem I.7.6, p. 494): i.e. it maps norm-bounded sets in C into weakly sequentially compact sets in C^*. By the Riesz representation theorem (for vector measures), there is a unique C^*-valued Borel measure λ on $[-r, 0]$ (of finite semi-variation) such that

$$\beta(\xi) = \int_{-r}^{0} \xi(s)\, d\lambda(s)$$

for all $\xi \in C$. ([D-S], Vol. I, Theorem VI.7.3, p. 493). By the dominated convergence theorem for vector measures ([D-S], Theorem IV.10.10, p. 328), one could reach elements in F_1 using weakly convergent sequences of type $\{\xi_0^k\}$. This gives a unique weakly continuous extension $\hat{\beta} : C \oplus F_1 \to C^*$. Next for each $\eta \in C$, $v \in \mathbf{R}$, extend $\hat{\beta}(\eta + v\chi_{\{0\}}) : C \to \mathbf{R}$ to a weakly continuous linear map $\overline{\hat{\beta}(\eta + v\chi_{\{0\}})} : C \oplus F_1 \to \mathbf{R}$. Thus $\overline{\beta}$ corresponds to the weakly continuous bilinear extension $\overline{\hat{\beta}(\cdot)}(\cdot) : [C \oplus F_1] \times [C \oplus F_1] \to \mathbf{R}$ of β. (Check this as exercise).

Finally use $\beta = D^2\phi(\eta)$ for each fixed $\eta \in C$ to get the required bilinear extension $\overline{D^2\phi(\eta)}$. $\square$

Lemma II.2. ([Mo], Pitman Books, 1984)

For $t > 0$ define $W_t^* \in C$ by

$$W_t^*(s) := \begin{cases} \dfrac{1}{\sqrt{t}}[W(t+s) - W(0)], & -t \le s < 0, \\ 0 & -r \le s \le -t. \end{cases}$$

Let β be a continuous bilinear form on C. Then

$$\lim_{t \to 0+} \left[\frac{1}{t} E\beta({}^{\eta}x_t - \tilde{\eta}_t, {}^{\eta}x_t - \tilde{\eta}_t) - E\beta(G(\eta) \circ W_t^*, G(\eta) \circ W_t^*) \right] = 0$$

Proof.

Use
$$\lim_{t \to 0+} E\| \frac{1}{\sqrt{t}}({}^{\eta}x_t - \tilde{\eta}_t - G(\eta) \circ W_t^* \|_C^2 = 0.$$
The above limit follows from the Lipschitz continuity of H and G and the martingale properties of the Itô integral. Conclusion of lemma is obtained by a computation using the bilinearity of β, Hölder's inequality and the above limit.([Mo], Pitman Books, 1984, pp. 86-87.)

$\square$

Lemma II.3. ([Mo], Pitman Books, 1984)

Let β be a continuous bilinear form on C and $\{e_i\}_{i=1}^m$ be any basis for $\mathbf{R}^m$. Then

$$\lim_{t \to 0+} \frac{1}{t} E\beta({}^{\eta}x_t - \tilde{\eta}_t, {}^{\eta}x_t - \tilde{\eta}_t) = \sum_{i=1}^{m} \overline{\beta}\big(G(\eta)(e_i)\chi_{\{0\}}, G(\eta)(e_i)\chi_{\{0\}}\big)$$

for each $\eta \in C$.

Proof.

By taking coordinates reduce to the one-dimensional case $d = m = 1$:

$$\lim_{t \to 0+} E\beta(W_t^*, W_t^*) = \overline{\beta}(\chi_{\{0\}}, \chi_{\{0\}})$$

with W one-dimensional Brownian motion. The proof of the above relation is lengthy and difficult. A key idea is the use of the projective tensor product $C \otimes_\pi C$ in order to view the continuous *bilinear* form β as a continuous *linear* functional on $C \otimes_\pi C$. At this level β commutes with the (Bochner) expectation. Rest of computation is effected using Mercer's theorem and some Fourier analysis. See [Mo], 1984, pp. 88-94. $\square$

Theorem II.3.([Mo], Pitman Books, 1984)

In (XIV) suppose H and G are globally bounded and Lipschitz. Let $S : D(S) \subset C_b \to C_b$ be the weak generator of $\{S_t\}$. Suppose $\phi \in D(S)$ is sufficiently smooth (e.g. ϕ is C^2, $D\phi$, $D^2\phi$ globally bounded and Lipschitz). Then $\phi \in D(A)$ and

$$\begin{aligned} A(\phi)(\eta) = S(\phi)(\eta) &+ \overline{D\phi(\eta)}\big(H(\eta)\chi_{\{0\}}\big) \\ &+ \frac{1}{2}\sum_{i=1}^{m} \overline{D^2\phi(\eta)}\big(G(\eta)(e_i)\chi_{\{0\}}, G(\eta)(e_i)\chi_{\{0\}}\big). \end{aligned}$$

where $\{e_i\}_{i=1}^m$ is any basis for $\mathbf{R}^m$.

Proof.

Step 1.

For fixed $\eta \in C$, use Taylor's theorem:

$$\phi({}^\eta x_t) - \phi(\eta) = \phi(\tilde{\eta}_t) - \phi(\eta) + D\phi(\tilde{\eta}_t)({}^\eta x_t - \tilde{\eta}_t) + R(t)$$

a.s. for $t > 0$; where

$$R(t) := \int_0^1 (1-u) D^2\phi[\tilde{\eta}_t + u({}^\eta x_t - \tilde{\eta}_t)]({}^\eta x_t - \tilde{\eta}_t, {}^\eta x_t - \tilde{\eta}_t)\, du.$$

Take expectations and divide by $t > 0$:

$$\left.\begin{aligned}\frac{1}{t}E[\phi({}^\eta x_t) - \phi(\eta)] = \frac{1}{t}[S_t(\phi(\eta){-}\phi(\eta)] + D\phi(\tilde{\eta}_t)\Big\{E[\frac{1}{t}({}^\eta x_t - \tilde{\eta}_t)]\Big\}\\ + \frac{1}{t}ER(t)\end{aligned}\right\} \tag{3}$$

for $t > 0$.

As $t \to 0+$, the first term on the RHS converges to $S(\phi)(\eta)$, because $\phi \in D(S)$.

Step 2.

Consider second term on the RHS of (3). Then

$$\begin{aligned}\lim_{t\to 0+}\left[E\left\{\frac{1}{t}({}^\eta x_t - \tilde{\eta}_t)\right\}\right](s) &= \begin{cases}\lim\limits_{t\to 0+}\dfrac{1}{t}\displaystyle\int_0^t E[H({}^\eta x_u)]\,du, & s = 0\\ 0 \qquad -r \le s < 0.\end{cases}\\ &= [H(\eta)\chi_{\{0\}}](s), \qquad\qquad -r \le s \le 0.\end{aligned}$$

Since H is bounded, then $\|E\{\frac{1}{t}({}^\eta x_t - \tilde{\eta}_t)\}\|_C$ is bounded in $t > 0$ and $\eta \in C$ (*Exercise*). Hence

$$w - \lim_{t\to 0+}\left[E\left\{\frac{1}{t}({}^\eta x_t - \tilde{\eta}_t)\right\}\right] = H(\eta)\chi_{\{0\}} \quad (\notin C).$$

Therefore by Lemma II.1 and the continuity of $D\phi$ at η:

$$\begin{aligned}\lim_{t\to 0+} D\phi(\tilde{\eta}_t)\left\{E\left[\frac{1}{t}({}^\eta x_t - \tilde{\eta}_t)\right]\right\} &= \lim_{t\to 0+} D\phi(\eta)\left\{E\left[\frac{1}{t}({}^\eta x_t - \tilde{\eta}_t)\right]\right\}\\ &= \overline{D\phi(\eta)}\big(H(\eta)\chi_{\{0\}}\big)\end{aligned}$$

Step 3.

To compute limit of third term in RHS of (3), consider

$$
\begin{aligned}
&\Big|\frac{1}{t}ED^2\phi[\tilde{\eta}_t + u({}^\eta x_t - \tilde{\eta}_t)]({}^\eta x_t - \tilde{\eta}_t, {}^\eta x_t - \tilde{\eta}_t) \\
&\quad - \frac{1}{t}ED^2\phi(\eta)({}^\eta x_t - \tilde{\eta}_t, {}^\eta x_t - \tilde{\eta}_t)\Big| \\
&\quad \leq (E\|D^2\phi[\tilde{\eta}_t + u({}^\eta x_t - \tilde{\eta}_t)] - D^2\phi(\eta)\|^2)^{1/2}\left[\frac{1}{t^2}E\|{}^\eta x_t - \tilde{\eta}_t\|^4\right]^{1/2} \\
&\quad \leq K(t^2+1)^{1/2}[E\|D^2\phi[\tilde{\eta}_t + u({}^\eta x_t - \tilde{\eta}_t)] - D^2\phi(\eta)\|^2]^{1/2} \\
&\quad \to 0
\end{aligned}
$$

as $t \to 0+$, uniformly for $u \in [0,1]$, by martingale properties of the Itô integral and the Lipschitz continuity of $D^2\phi$. Therefore by Lemma II.3

$$
\begin{aligned}
\lim_{t\to 0+}\frac{1}{t}ER(t) &= \int_0^1 (1-u)\lim_{t\to 0+}\frac{1}{t}ED^2\phi(\eta)({}^\eta x_t - \tilde{\eta}_t, {}^\eta x_t - \tilde{\eta}_t)\,du \\
&= \frac{1}{2}\sum_{i=1}^m \overline{D^2\phi(\eta)}\big(G(\eta)(e_i)\chi_{\{0\}}, G(\eta)(e_i)\chi_{\{0\}}\big).
\end{aligned}
$$

The above is a weak limit since $\phi \in D(S)$ and has first and second derivatives globally bounded on C. $\square$

5. Quasitame Functions

Recall that a function $\phi : C \to \mathbf{R}$ is *tame* (or a *cylinder function*) if there is a finite set $\{s_1 < s_2 < \cdots < s_k\}$ in $[-r, 0]$ and a C^∞-bounded function $f : (\mathbf{R}^d)^k \to \mathbf{R}$ such that

$$\phi(\eta) = f(\eta(s_1), \cdots, \eta(s_k)), \qquad \eta \in C.$$

The set of all tame functions is a weakly dense subalgebra of C_b, invariant under the static shift S_t and generates $Borel\, C$. For $k \geq 2$ the tame function ϕ *lies outside* the domain of strong continuity C_b^0 of P_t, and hence *outside* $D(A)$ ([Mo], Pitman Books, 1984, pp.98-103; see also proof of Theorem IV .2.2, pp. 73-76). To overcome this difficulty we introduce

Definition.

Say $\phi : C \to \mathbf{R}$ is *quasitame* if there are C^∞-bounded maps $h : (\mathbf{R}^d)^k \to \mathbf{R}$, $f_j : \mathbf{R}^d \to \mathbf{R}^d$, and piecewise C^1 functions $g_j : [-r, 0] \to \mathbf{R}, 1 \le j \le k-1$, such that

$$\phi(\eta) = h\Big(\int_{-r}^{0} f_1(\eta(s))g_1(s)\,ds, \cdots, \int_{-r}^{0} f_{k-1}(\eta(s))g_{k-1}(s)\,ds, \eta(0)\Big) \tag{4}$$

for all $\eta \in C$.

Theorem II.4. ([Mo], Pitman Books, 1984)

The set of all quasitame functions is a weakly dense subalgebra of C_b^0, invariant under S_t, generates $Borel\, C$ and belongs to $D(A)$. In particular, if ϕ is the quasitame function given by (4), then

$$\begin{aligned} A(\phi)(\eta) = \sum_{j=1}^{k-1} & D_j h(m(\eta))\{f_j(\eta(0))g_j(0) - f_j(\eta(-r))g_j(-r) \\ & - \int_{-r}^{0} f_j(\eta(s))g_j'(s)\,ds\} \\ & + D_k h(m(\eta))(H(\eta)) + \frac{1}{2}\text{trace}[D_k^2 h(m(\eta)) \circ (G(\eta) \times G(\eta))]. \end{aligned}$$

for all $\eta \in C$, where

$$m(\eta) := \Big(\int_{-r}^{0} f_1(\eta(s))g_1(s)\,ds, \cdots, \int_{-r}^{0} f_{k-1}(\eta(s))g_{k-1}(s)\,ds, \eta(0)\Big).$$

Remarks.

(i) Replace C by the Hilbert space M_2. No need for the weak extensions because M_2 is weakly complete. Extensions of $D\phi(v, \eta)$ and $D^2\phi(v, \eta)$ correspond to partial derivatives in the $\mathbf{R}^d$-variable. *Tame functions do not exist on* M_2 but quasitame functions do! (with $\eta(0)$ replaced by $v \in \mathbf{R}^d$).

Analysis of supermartingale behavior and stability of $\phi({}^{\eta}x_t)$ given in Kushner ([Ku], JDE, 1968). Infinite fading memory setting by Mizel and Trützer ([M-T], JIE, 1984) in the weighted state space $\mathbf{R}^d \times L^2((-\infty, 0], \mathbf{R}^d; \rho)$.

(ii) For each quasitame ϕ on C, $\phi({}^{\eta}x_t)$ *is* a semimartingale, and the Itô formula holds:

$$d[\phi({}^{\eta}x_t)] = A(\phi)({}^{\eta}x_t)\,dt + \overline{D\phi(\eta)}\big(H(\eta)\chi_{\{0\}}\big)\,dW(t).$$

III. REGULARITY
CLASSIFICATION OF SFDE'S

Geilo, Norway

Wednesday, July 31, 1996

10:30-11:20

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III. REGULARITY. CLASSIFICATION OF SFDE'S

Denote the state space by E where $E = C$ or $M_2 := \mathbf{R}^d \times L^2([-r,0],\mathbf{R}^d)$. Most results hold for either choice of state space.

Objectives

To study regularity properties of the trajectory of a sfde as a random field $X := \{{}^\eta x_t : t \geq 0, \eta \in C\}$ in the variables (t,η,ω) $\big(E = C\big)$ or $(t,(v,\eta),\omega)$ $\big(E = M_2\big)$:

(i) Pathwise regularity of trajectories in the time variable.

(ii) Regularity of trajectories (in probability or pathwise) in the initial state $\eta \in C$ or $(v,\eta) \in M_2$.

(iii) Classification of sfde's into regular and singular types.

Denote by $C^\alpha := C^\alpha([-r,0],\mathbf{R}^d)$ the (separable) Banach space of α-Hölder continuous paths $\eta : [-r,0] \to \mathbf{R}^d$ with the Hölder norm

$$\|\eta\|_\alpha := \|\eta\|_C + \sup\left\{\frac{|\eta(s_1)-\eta(s_2)|}{|s_1-s_2|^\alpha} : s_1, s_2 \in [-r,0], s_1 \neq s_2\right\}.$$

C^α can be constructed in a *separable manner* by completing the space of smooth paths $[-r,0] \to \mathbf{R}^d$ with respect to the above norm (Tromba [Tr], JFA, 1972). First step is to think of ${}^\eta x_t(\omega)$ as a measurable mapping $X : \mathbf{R}^+ \times C \times \Omega \to C$ in the three variables (t,η,ω) simultaneously:

Theorem III.1([Mo], Pitman Books, 1984)

In the sfde

$$\left.\begin{aligned} dx(t) &= H(t,x_t)\,dt + G(t,x_t)\,dW(t) \quad t > 0 \\ x_0 &= \eta \in C \end{aligned}\right\} \qquad (XIII)$$

assume that the coefficients H, G are (jointly) continuous and globally Lipschitz in the second variable uniformly wrt the first. Then *(i) For any* $0 < \alpha < \dfrac{1}{2}$, *and each initial path* $\eta \in C$,

$$P({}^{\eta}x_t \in C^{\alpha}, \textit{for all} \ \ t \geq r) = 1.$$

(ii) the trajectory field has a measurable version

$$X : \mathbf{R}^{+} \times C \times \Omega \to C.$$

(iii) The trajectory field ${}^{\eta}x_t, t \geq r,\ \eta \in C$, *admits a measurable version*

$$[r, \infty) \times C \times \Omega \to C^{\alpha}.$$

Remark.

Similar statements hold for $E = M_2$.

Give $L^0(\Omega, E)$ the complete (psuedo)metric

$$d_E(\theta_1, \theta_2) := \inf_{\epsilon > 0}[\epsilon + P(\|\theta_1 - \theta_2\|_E \geq \epsilon)], \quad \theta_1, \theta_2 \in L^0(\Omega, E),$$

(which corresponds to convergence in probability, Dunford and Schwartz [D-S], Lemma III.2.7, p. 104).

Proof of Theorem III.1.

(i) Sufficient to show that

$$P\big({}^{\eta}x|[0,a] \in C^{\alpha}([0,a], \mathbf{R}^d)\big) = 1$$

by using the estimate

$$P\left(\sup_{0 \leq t_1, t_2 \leq a, t_1 \neq t_1} \frac{|{}^{\eta}x(t_1) - {}^{\eta}x(t_2)|}{|t_1 - t_2|^{\alpha}} \geq N\right) \leq C_k^1(1 + \|\eta\|_C^{2k})\frac{1}{N^{2k}},$$

for all integers $k > (1 - 2\alpha)^{-1}$, and the Borel-Cantelli lemma. Above estimate is proved using Gronwall's lemma, Chebyshev's inequality, and Garsia-Rodemick-Rumsey lemma ([Mo], Pitman

Books, 1984, Theorem 4.1, p. 150; [Mo], Pitman Books, 1984, Theorem 4.4, pp.152-154.)

(ii) By mean-square Lipschitz dependence ([Mo], Pitman Books, 1984, Theorem 3.1, p. 41), the trajectory

$$\begin{aligned} [0,a] \times C &\to L^2(\Omega, C) \subset L^0(\Omega, C) \\ (t,\eta) &\mapsto {}^{\eta}x_t \end{aligned}$$

is globally Lipschitz in η uniformly wrt t in compact sets, and is continuous in t for fixed η. Therefore it is jointly continuous in (t,η) as a map

$$[0,a] \times C \ni (t,\eta) \mapsto {}^{\eta}x_t \in L^0(\Omega, C).$$

Then apply the Cohn-Hoffman-Jϕrgensen theorem:
If T, E are complete separable metric spaces, then each Borel map $X : T \to L^0(\Omega, E; \mathcal{F})$ admits a measurable version

$$T \times \Omega \to E$$

to the trajectory field to get measurability in (t,η). (Take $T = [0,a] \times C$, $E = C$ ([Mo], Pitman Books, 1984, p. 16).)

(iii) Use the estimate

$$P\big(\|{}^{\eta_1}x_t - {}^{\eta_2}x_t\|_{C^\alpha} \geq N\big) \leq \frac{C_k^2}{N^{2k}}\|\eta_1 - \eta_2\|_C^{2k}$$

for $t \in [r,a], N > 0$, ([Mo], 1984, Theorem 4.7, pp.158-162) to prove joint continuity of the trajectory

$$\begin{aligned} [r,a] \times C &\to L^0(\Omega, C^\alpha) \\ (t,\eta) &\mapsto {}^{\eta}x_t \end{aligned}$$

([Mo], Theorem 4.7, pp. 158-162) viewed as a process with values in the separable Banach space C^α. Again apply the Cohn-Hoffman-Jϕrgensen theorem. $\square$

As we have seen in Lecture I, the trajectory of a sfde possesses good regularity properties *in the mean-square.* The following theorem shows good behavior in distribution.

Theorem III.2. ([Mo], Pitman Books, 1984)

Suppose the coefficients H, G are globally Lipschitz in the second variable uniformly with respect to the first. Let $\alpha \in (0, 1/2)$ and k be any integer such that $k > (1-2\alpha)^{-1}$. Then there are positive constants C_k^3, C_k^4, C_k^5 such that

$$d_C({}^{\eta_1}x_t, {}^{\eta_2}x_t) \leq C_k^3 \|\eta_1 - \eta_2\|_C^{2k/(2k+1)} \qquad t \in [0, a]$$

$$d_{C^\alpha}({}^{\eta_1}x_t, {}^{\eta_2}x_t) \leq C_k^4 \|\eta_1 - \eta_2\|_C^{2k/(2k+1)} \qquad t \in [r, a]$$

$$P\Big(\|{}^{\eta}x_t\|_{C^\alpha} \geq N\Big) \leq C_k^5 (1 + \|\eta\|_C^{2k}) \frac{1}{N^{2k}}, \qquad t \in [r, a], \quad N > 0.$$

In particular the transition probabilities

$$\begin{aligned} [r, a] \times C &\to \mathcal{M}_p(C) \\ (t,\eta) &\mapsto p(0, \eta, t, \cdot) \end{aligned}$$

take bounded sets into relatively weak compact sets in the space $\mathcal{M}_p(C)$ of probability measures on C.*

Proof of Theorem III.2.

Proofs of the estimates use Gronwall's lemma, Chebyshev's inequality, and Garsia-Rodemick-Rumsey lemma ([Mo], 1984, Theorem 4.1, p. 150; [Mo], 1984, Theorem 4.7, pp.159-162.) The weak* compactness assertion follows from the last estimate, Prohorov's theorem and the compactness of the embedding $C^\alpha \hookrightarrow C$ ([Mo], 1984, Theorem 4.6, pp. 156-158). $\square$

Erratic Behavior. The Noisy Loop Revisited

Definition.

A sfde is *regular* with respect to M_2 if its trajectory random field $\{(x(t), x_t) : (x(0), x_0) = (v, \eta) \in M_2,\ t \geq 0\}$ admits a $(Borel\, \mathbf{R}^+ \otimes Borel\, M_2 \otimes \mathcal{F},\ Borel\, M_2)$-measurable version $X : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ with a.a. sample functions continuous on $\mathbf{R}^+ \times M_2$. The sfde is said to be *singular* otherwise. Similarly for regularity with respect to C.

Consider the one-dimensional linear sdde with a *positive delay* r

$$\left.\begin{aligned} dx(t) &= \sigma x(t-r))\, dW(t), \quad t > 0 \\ (x(0), x_0) &= (v, \eta) \in M_2 := \mathbf{R} \times L^2([-r, 0], \mathbf{R}), \end{aligned}\right\} \qquad (I)$$

driven by a Wiener process W.

Theorem III.3 below implies that (I) is singular with respect to M_2 (and C).(See also [Mo], Stochastics, 1986).

Consider the regularity of the more general one-dimen-sional linear sfde:

$$\left.\begin{aligned} dx(t) &= \int_{-r}^{0} x(t+s) d\nu(s)\, dW(t), \quad t > 0 \\ (x(0), x_0) &\in M_2 := \mathbf{R} \times L^2([-r, 0], \mathbf{R}) \end{aligned}\right\} \qquad (II')$$

where W is a Wiener process and ν is a fixed finite real-valued Borel measure on $[-r, 0]$.

Exercise:

(II$'$) is regular if ν has a C^1 (or even L_1^2) density with respect to Lebesgue measure on $[-r, 0]$. (Hint: Use integration by parts to eliminate the Itô integral!)

The following theorem gives conditions on the measure ν under which (II$'$) is singular.

Theorem III.3 ([M-S], II, 1996)

Let $r > 0$, and suppose that there exists $\epsilon \in (0, r)$ such that $\operatorname{supp} \nu \subset [-r, -\epsilon]$. *Suppose $0 < t_0 \le \epsilon$. For each $k \ge 1$, set*

$$\nu_k := \sqrt{t_0} \left| \int_{[-r,0]} e^{2\pi i k s / t_0} \, d\nu(s) \right|.$$

Assume that

$$\sum_{k=1}^{\infty} \nu_k x^{1/\nu_k^2} = \infty \tag{1}$$

for all $x \in (0, 1)$. Let $Y : [0, \epsilon] \times M_2 \times \Omega \to \mathbf{R}$ be any Borel-measurable version of the solution field $\{x(t) : 0 \le t \le \epsilon, \ (x(0), x_0) = (v, \eta) \in M_2\}$ of (II$'$). Then for a.a. $\omega \in \Omega$, the map $Y(t_0, \cdot, \omega) : M_2 \to \mathbf{R}$ is unbounded in every neighborhood of every point in M_2, and (hence) non-linear.

Corollary. ([Mo], Pitman Books, 1984)

Suppose $r > 0, \sigma \neq 0$ *in (I). Then the trajectory* $\{{}^\eta x_t : 0 \leq t \leq r, \eta \in C\}$ *of (I) has a measurable version* $X : \mathbf{R}^+ \times C \times \Omega \to C$ *s.t. for every* $t \in (0, r]$

$$P\Big(X(t, \eta_1 + \lambda\eta_2, \cdot) = X(t, \eta_1, \cdot) + \lambda X(t, \eta_2, \cdot) \quad \text{for all } \lambda \in \mathbf{R}, \eta_1, \eta_2 \in C \Big) = 0.$$

But

$$P\Big(X(t, \eta_1 + \lambda\eta_2, \cdot) = X(t, \eta_1, \cdot) + \lambda X(t, \eta_2, \cdot) \Big) = 1.$$

for all $\lambda \in \mathbf{R}, \eta_1, \eta_2 \in C$.

Remark.

(i) Condition (1) of the theorem is implied by

$$\lim_{k \to \infty} \nu_k \sqrt{\log k} = \infty.$$

(ii) For the delay equation (I), $\nu = \sigma\delta_{-r},\ \epsilon = r$. In this case condition (1) is satisfied for *every* $t_0 \in (0, r]$.

(iii) Theorem III.3 also holds for state space C since every bound-ed set in C is also bounded in $L^2([-r, 0], \mathbf{R})$.

Proof of Theorem III.3.

Joint work with V. Mizel.

Main idea is to track the solution random field of (a complexified version of) (II′) along the classical Fourier basis

$$\eta_k(s) = \mathrm{e}^{2\pi iks/t_0} \quad , \qquad -r \le s \le 0, \qquad k \ge 1 \tag{2}$$

in $L^2([-r,0],\mathbf{C})$. On this basis, the solution field gives an infinite family of independent Gaussian random variables. This allows us to show that no Borel measurable version of the solution field can be bounded with positive probability on an arbitrarily small neighborhood of 0 in M_2, and hence on any neighborhood of any point in M_2 (cf. [Mo], Pitman Books, 1984; [Mo], Stochastics, 1986). For simplicity of computations, complexify the state space in (II′) by allowing (v,η) to belong to $M_2^C := \mathbf{C} \times L^2([-r,0],\mathbf{C})$. Thus consider the sfde

$$\left.\begin{aligned} dx(t) &= \int_{[-r,0]} x(t+s)d\nu(s)\, dW(t), t>0, \\ (x(0),x_0) &= (v,\eta) \in M_2^C \end{aligned}\right\} \qquad (II'-C))$$

where $x(t) \in \mathbf{C}$, $t \ge -r$, and ν, W are real-valued.

Use contradiction. Let $Y : [0,\epsilon] \times M_2 \times \Omega \to \mathbf{R}$ be any Borel-measurable version of the solution field $\{x(t) : 0 \le t \le \epsilon,\ (x(0),x_0) = (v,\eta) \in M_2\}$ of (II′). Suppose, if possible, that there exists a set $\Omega_0 \in \mathcal{F}$ of positive P-measure, $(v_0,\eta_0) \in M_2$ and a positive δ such that for all $\omega \in \Omega_0$, $Y(t_0,\cdot,\omega)$ is bounded on the open ball $B((v_0,\eta_0),\delta)$ in M_2 of center (v_0,η_0) and radius δ. Define the complexification $Z(\cdot,\omega) : M_2^C \to \mathbf{C}$ of $Y(t_0,\cdot,\omega) : M_2 \to \mathbf{R}$ by

$$Z(\xi_1 + i\xi_2,\omega) := Y(t_0,\xi_1,\omega) + i\, Y(t_0,\xi_2,\omega), \qquad i = \sqrt{-1},$$

for all $\xi_1, \xi_2 \in M_2$, $\omega \in \Omega$. Let $(v_0,\eta_0)^C$ denote the complexification $(v_0,\eta_0)^C := (v_0,\eta_0) + i(v_0,\eta_0)$. Clearly $Z(\cdot,\omega)$ is bounded on the complex ball $B((v_0,\eta_0)^C,\delta)$ in M_2^C for all $\omega\in\Omega_0$. Define the sequence of complex random variables $\{Z_k\}_{k=1}^\infty$ by

$$Z_k(\omega):=Z((\eta_k(0),\eta_k),\omega)-\eta_k(0), \qquad \omega\in\Omega, \quad k\geq 1.$$

Then

$$Z_k=\int_0^{t_0}\int_{[-r,-\epsilon]}\eta_k(u+s)\,d\nu(s)\,dW(u), \quad k\geq 1.$$

By standard properties of the Itô integral, and Fubini's theorem,

$$EZ_k\overline{Z_l}=\int_{[-r,-\epsilon]}\int_{[-r,-\epsilon]}\int_0^{t_0}\eta_k(u+s)\overline{\eta_l(u+s')}\,du\,d\nu(s)\,d\nu(s')=0$$

for $k\neq l$, because

$$\int_0^{t_0}\eta_k(u+s)\overline{\eta_l(u+s')}\ du=0$$

whenever $k\neq l$, for all $s,s'\in[-r,0]$. Furthermore

$$\int_0^{t_0}\eta_k(u+s)\overline{\eta_k(u+s')}\ du=t_0\mathrm{e}^{2\pi ik(s-s')/t_0}$$

for all $s,s'\in[-r,0]$. Hence

$$\begin{aligned}E|Z_k|^2&=\int_{[-r,-\epsilon]}\int_{[-r,-\epsilon]}t_0\mathrm{e}^{2\pi ik(s-s')/t_0}\ d\nu(s)\,d\nu(s')\\&=t_0\left|\int_{[-r,0]}\mathrm{e}^{2\pi iks/t_0}\ d\nu(s)\right|^2\\&=\nu_k^2.\end{aligned}$$

$Z(\cdot,\omega):M_2^C\to\mathbf{C}$ is bounded on $B((v_0,\eta_0)^C,\delta)$ for all $\omega\in\Omega_0$, and $\|(\eta_k(0),\eta_k)\|=\sqrt{r+1}$ for all $k\geq 1$. By the linearity property

$$\begin{aligned}&Z\left((v_0,\eta_0)^C+\frac{\delta}{2\sqrt{r+1}}(\eta_k(0),\eta_k),\cdot\right)\\&\qquad=Z((v_0,\eta_0)^C,\cdot)+\frac{\delta}{2\sqrt{r+1}}Z((\eta_k(0),\eta_k),\cdot),\ k\geq 1,\end{aligned}$$

a.s., it follows that

$$P\bigl(\sup_{k\geq 1}|Z_k| < \infty\bigr) > 0. \tag{3}$$

It is easy to check that $\{ReZ_k,\, ImZ_k : k \geq 1\}$ are independent $\mathcal{N}(0, \nu_k^2/2)$-distributed Gaussian random variables. Get a contradiction to (3):

For each integer $N \geq 1$,

$$\begin{aligned} P\biggl(\sup_{k\geq 1}|Z_k| < N\biggr) &\leq \prod_{k\geq 1} P\biggl(|ReZ_k| < N\biggr) \\ &= \prod_{k\geq 1}\left[1 - \frac{2}{\sqrt{2\pi}}\int_{\frac{\sqrt{2}N}{\nu_k}}^{\infty} \mathrm{e}^{-x^2/2}\,dx\right] \\ &\leq \exp\biggl\{-\frac{2}{\sqrt{2\pi}}\sum_{k=1}^{\infty}\int_{\frac{\sqrt{2}N}{\nu_k}}^{\infty} \mathrm{e}^{-x^2/2}\,dx\biggr\}. \end{aligned} \tag{4}$$

There exists $N_0 > 1$ (independent of $k \geq 1$) such that

$$\int_{\frac{\sqrt{2}N}{\nu_k}}^{\infty} \mathrm{e}^{-x^2/2}\,dx \geq \frac{\nu_k}{2\sqrt{2}N}\mathrm{e}^{-\frac{N^2}{\nu_k^2}} \tag{5}$$

for all $N \geq N_0$ and all $k \geq 1$.

Combine (4) and (5) and use hypothesis (1) of the theorem to get

$$P\biggl(\sup_{k\geq 1}|Z_k| < N\biggr) = 0$$

for all $N \geq N_0$. Hence

$$P\biggl(\sup_{k\geq 1}|Z_k| < \infty\biggr) = 0.$$

This contradicts (3)(cf. Dudley [Du], JFA, 1967).

Since $Y(t_0, \cdot, \omega)$ is locally unbounded, it must be non-linear because of Douady's Theorem:

Every Borel measurable linear map between two Banach spaces is continuous.

(Schwartz [Sc], Radon Measures, Part II, 1973, pp. 155-160). $\square$

Note that the pathological phenomenon in Theorem III.3 is peculiar to the delay case $r > 0$. The proof of the theorem suggests that this pathology is due to the *Gaussian nature* of the Wiener process W coupled with the *infinite-dimensionality* of the state space M_2. Because of this, one may expect similar difficulties in certain types of linear spde's driven by *multi-dimensional* white noise (Flandoli and Schaumlöffel [F-S], Stochastics, 1990).

Problem.

Classify all finite signed measures ν on $[-r, 0]$ for which (II') is regular.

Note that (I) automatically satisfies the conditions of Theorem III.3, and hence its trajectory field *explodes on every small neighborhood of* $0 \in M_2$. Because of the singular nature of (I), it is surprising that the maximal exponential growth rate of the trajectory of (I) is *negative* for small σ and is bounded away from zero *independently of the choice of the initial path* in M_2. This will be shown later in Lecture V (Theorem V.1).

Regular Linear Systems. White Noise

SDE's on $\mathbf{R}^d$ driven by m-dimensional Brownian motion $W := (W_1, \cdots, W_m)$, with smooth coefficients.

$$\left.\begin{aligned} dx(t) &= H(x(t-d_1), \cdots, x(t-d_N), x(t), x_t)dt \\ &\qquad + \sum_{i=1}^{m} g_i x(t)\, dW_i(t), \qquad t > 0 \\ (x(0), x_0) &= (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d) \end{aligned}\right\} \qquad (VIII)$$

(VIII) is defined on

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P) =$ canonical complete filtered Wiener space:

$\Omega :=$ space of all continuous paths $\omega : \mathbf{R}^+ \to \mathbf{R}^m$, $\omega(0) = 0$, in Euclidean space $\mathbf{R}^m$, with compact open topology;

$\mathcal{F} :=$ completed Borel σ-field of Ω;

$\mathcal{F}_t :=$ completed sub-σ-field of $\mathcal{F}$ generated by the evaluations $\omega \to \omega(u)$, $0 \le u \le t$, $\quad t \ge 0$;

$P :=$ Wiener measure on Ω;

$dW_i(t) =$ Itô stochastic differentials, $1 \le i \le m$.

Several finite delays $0 < d_1 < d_2 < \cdots < d_N \le r$ in drift term; *no delays in diffusion coefficient.*

$H : (\mathbf{R}^d)^{N+1} \times L^2([-r,0],\mathbf{R}^d) \to \mathbf{R}^d$ is a fixed continuous linear map; g_i, $i = 1, 2, \ldots, m$, fixed (deterministic) $d \times d$-matrices.

Theorem III.4.([Mo], Stochastics, 1990])

(VIII) is regular with respect to the state space $M_2 = \mathbf{R}^d \times \mathbf{L}^2([-r,0],\mathbf{R}^d)$. There is a measurable version $X : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ of the trajectory field $\{(x(t), x_t) : t \in \mathbf{R}^+,\ (x(0), x_0) = (v,\eta) \in M_2\}$ with the following properties:

(i) For each $(v,\eta) \in M_2$ and $t \in \mathbf{R}^+$, $X(t,(v,\eta),\cdot) = (x(t), x_t)$ a.s., is $\mathcal{F}_t$-measurable and belongs to $L^2(\Omega, M_2; P)$.

(ii) There exists $\Omega_0 \in \mathcal{F}$ of full measure such that, for all $\omega \in \Omega_0$, the map $X(\cdot,\cdot,\omega) : \mathbf{R}^+ \times M_2 \to M_2$ is continuous.

(iii) For each $t \in \mathbf{R}^+$ and every $\omega \in \Omega_0$, the map $X(t,\cdot,\omega) : M_2 \to M_2$ is continuous linear; for each $\omega \in \Omega_0$, the map $\mathbf{R}^+ \ni t \mapsto X(t,\cdot,\omega) \in L(M_2)$ is measurable and locally bounded in the uniform operator norm on $L(M_2)$. The map $[r,\infty) \ni t \mapsto X(t,\cdot,\omega) \in L(M_2)$ is continuous for all $\omega \in \Omega_0$.

(iv) For each $t \ge r$ and all $\omega \in \Omega_0$, the map

$$X(t,\cdot,\omega) : M_2 \to M_2$$

is compact.

Proof uses variational technique to reduce the problem to the solution of a random family of classical integral equations involving *no stochastic integrals.*

Compactness of semi-flow for $t \geq r$ will be used later to define hyperbolicity for (VIII) and the associated exponential dichotomies (Lecture IV).

Regular Linear Systems. Semimartingale Noise

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ a complete filtered probability space satisfying the usual conditions.

Linear systems driven by semimartingale noise, and memory driven by a measure-valued process
$\nu : \mathbf{R} \times \Omega \to \mathcal{M}([-r,0], \mathbf{R}^{d\times d})$, where $\mathcal{M}([-r,0], \mathbf{R}^{d\times d})$ is the space of all $d \times d$-matrix-valued Borel measures on $[-r,0]$ (or $R^{d\times d}$-valued functions of bounded variation on $[-r,0]$). This space is given the σ-algebra generated by all evaluations. The space $\mathbf{R}^{d\times d}$ of all $d \times d$-matrices is given the Euclidean norm $\|\cdot\|$.

$$\left.\begin{aligned} dx(t) = \Big\{\int_{[-r,0]} \nu(t)(ds)\, x(t+s)\Big\}\, dt + dN(t) \int_{-r}^{0} K(t)(s)\, x(t+s)\, ds \\ + dL(t)\, x(t-), \qquad t > 0 \\ (x(0), x_0) = (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r,0], \mathbf{R}^d) \end{aligned}\right\} \qquad (IX)$$

Hypotheses (R)

(i) The process $\nu : \mathbf{R}\times\Omega \to \mathcal{M}([-r,0], \mathbf{R}^{d\times d})$ is measurable and $(\mathcal{F}_t)_{t\geq 0}$)-adapted. For each $\omega \in \Omega$ and $t \geq 0$ define the positive measure $\bar{\nu}(t,\omega)$ on $[-r,\infty)$ by

$$\bar{\nu}(t,\omega)(A) := |\nu|(t,\omega)\{(A-t) \cap [-r,0]\}$$

for all Borel subsets A of $[-r, \infty)$, where $|\nu|$ is the total variation measure of ν wrt the Euclidean norm on $\mathbf{R}^{d\times d}$. Therefore the equation

$$\mu(\omega)(\cdot) := \int_0^\infty \bar{\nu}(t,\omega)(\cdot)\, dt$$

defines a positive measure on $[-r, \infty)$. For each $\omega \in \Omega$ suppose that $\mu(\omega)$ has a density wrt Lebesgue measure which is locally essentially bounded.
(*Exercise:* This condition is automatically satisfied if $\nu(t,\omega)$ is independent of (t,ω).)

(ii) $K : \mathbf{R} \times \Omega \to L^\infty([-r, o], \mathbf{R}^{d\times d})$ is measurable and $(\mathcal{F}_t)_{t\geq 0}$- adapted. Define the random field $\tilde{K}(t,s,\omega)$ by
$\tilde{K}(t,s,\omega) := K(t,\omega)(s-t)$ for $t \geq 0$, $-r \leq s-t \leq 0$. Assume that $\tilde{K}(t,s,\omega)$ is absolutely continuous in t for Lebesgue a.a. s and all $\omega \in \Omega$. For every $\omega \in \Omega$, $\dfrac{\partial \tilde{K}}{\partial t}(t,s,\omega)$ and $\tilde{K}(t,s,\omega)$ are locally essentially bounded in (t,s). $\dfrac{\partial \tilde{K}}{\partial t}(t,s,\omega)$ is jointly measurable.

(iii) $L = M + V$, M continuos local martingale, V B.V. process.

Theorem III.5. ([M-S], I, AIHP, 1996)

Under hypotheses (R), equation (IX) is regular w.r.t. M_2 with a measurable flow $X : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$. This flow satisfies Theorem III.4.

Proof.

This is achieved via a construction in ([M-S], I, AIHP, 1996) which reduces (IX) to a random linear integral equation with *no stochastic integrals* ([M-S], AIHP, 1996, pp. 85-96). Do a complicated pathwise analysis on the integral equation to establish existence and regularity properties of the semiflow. □

Regular Non-linear Systems

(a) SFDE's with Ordinary Diffusion Coefficients

In the sfde,

$$\left.\begin{aligned} dx(t) &= H(x_t)dt + \sum_{i=1}^{m} g_i(x(t))dW_i(t) \\ x_0 &= \eta \in C \end{aligned}\right\} \qquad (XV)$$

let $H : C \to \mathbf{R}^d$ be globally Lipschitz and $g_i : \mathbf{R}^d \to \mathbf{R}^d$ C^2-bounded maps satisfying the Frobenius condition (vanishing Lie brackets):

$$Dg_i(v)g_j(v) = Dg_j(v)g_i(v), \quad 1 \le i, j \le m, \ v \in \mathbf{R}^d;$$

and $W := (W_1, W_2, \cdots, W_m)$ is m-dimensional Brownian motion. Note that the diffusion coefficient in (XV) has no memory.

Theorem III.6 ([Mo], Pitman Books, 1984)

Suppose the above conditions hold. Then the trajectory field $\{{}^\eta x_t : t \ge 0, \eta \in C\}$ *of (XV) has a measurable version* $X : \mathbf{R}^+ \times C \times \Omega \to C$ *satisfying the following properties. For each* $\alpha \in (0, 1/2)$, *there is a set* $\Omega_\alpha \subset \Omega$ *of full measure such that for every* $\omega \in \Omega_\alpha$

(i) $X(\cdot, \cdot, \omega) : \mathbf{R}^+ \times C \to C$ *is continuous;*

(ii) $X(\cdot, \cdot, \omega) : [r, \infty) \times C \to C^\alpha$ *is continuous;*

(iii) for each $t \ge r$, $X(t, \cdot, \omega) : C \to C$ *is compact;*

(iv) for each $t \ge r$, $X(t, \cdot, \omega) : C \to C^\alpha$ *is Lipschitz on every bounded set in* C, *with a Lipschitz constant independent of* t *in compact sets. Hence each map* $X(t, \cdot, \omega) : C \to C$ *is compact: viz. takes bounded sets into relatively compact sets.*

Proof of Theorem III.6.

([Mo], Pitman Books, 1984, Theorem (2.1), Chapter (V), §2, p. 121). This latter result is proved using a non-linear variational method originally due to Sussman ([Su], Ann. Prob., 1978) and Doss ([Do], AIHP, 1977) in the non-delay case $r = 0$. Write $g := (g_1, g_2, \cdots, g_m) : \mathbf{R}^d \to \mathbf{R}^{d\times m}$. By the Frobenius condition, there is a C^2 map $F : \mathbf{R}^m \times \mathbf{R}^d \to \mathbf{R}^d$ such that $\{F(\underline{t}, \cdot) : \underline{t} \in \mathbf{R}^m\}$ is a group of C^2 diffeomorphisms $\mathbf{R}^d \to \mathbf{R}^d$ satisfying

$$\begin{aligned} D_1 F(\underline{t}, x) &= g(F(\underline{t}, x)), \\ F(\underline{0}, x) &= x \end{aligned}$$

for all $\underline{t} \in \mathbf{R}^m, x \in \mathbf{R}^d$.

Define

$$W^0(t) := \begin{cases} W(t) - W(0), & t \geq 0 \\ 0 & -r \leq t < 0 \end{cases}$$

and $\tilde{H} : \mathbf{R}^+ \times C \times \Omega \to \mathbf{R}^d$, by

$$\begin{aligned} \tilde{H}(t, \eta, \cdot) :=& D_2 F(W^0(t), \eta(0))^{-1} \big\{ H[F \circ (W_t^0, \eta)] \\ & - \frac{1}{2} \text{trace}\big(Dg[F(W^0(t), \eta(0))] \circ g[F(W^0(t), \eta(0))] \big) \big\} \end{aligned}$$

where the expression under the "trace" is viewed as a bilinear form $\mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}^d$, and the trace has values in $\mathbf{R}^d$. Then for each ω, $\tilde{H}(t, \eta, \omega)$ is jointly continuous, Lipschitz in η in bounded subsets of C uniformly for t in compact sets, and satisfies a global linear growth condition in η ([Mo], Pitman Books, 1984, pp. 114-126).

Therefore solve the fde

$$\begin{aligned} &{}^{\eta}\xi_t' = \tilde{H}(t, {}^{\eta}\xi_t, \cdot) \qquad t \geq 0 \\ &{}^{\eta}\xi_0 = \eta. \end{aligned}$$

Define the semiflow

$$X(t, \eta, \omega) = F \circ \big(W_t^0(\omega), {}^{\eta}x_t(\omega)\big).$$

Check that X satisfies all assertions of theorem ([Mo], 1984, pp.126-133). $\square$

(b) SFDE's with Smooth Memory

$$\left.\begin{aligned} dx(t) &= H(dt, x(t), x_t) + G(dt, x(t), g(x_t)), \quad t > 0 \\ (x(0), x_0) &= (v, eta) \in M_2 \end{aligned}\right\} \qquad (XVI)$$

Coefficients H and G in (XVI) are semimartingale-valued random fields on $M_2 = \mathbf{R}^d \times L^2([-r,0], \mathbf{R}^d)$ and $\mathbf{R}^d \times \mathbf{R}^m$, respectively. The memory is driven by a functional $g : L^2([-r,0], \mathbf{R}^d) \to \mathbf{R}^m$ with the smoothness property that the process $t \mapsto g(x_t)$ has absolutely continuous paths for each adapted process x. Under (technical) but general regularity and boundedness conditions on the characteristics of H and G, equation (XVI) is regular:

Theorem III.7 ([M-S], 1996)

Let

$$\Delta := \{(t_0, t) \in \mathbf{R}^2 : t_0 \le t\}.$$

Under suitable regularity conditions on H, G, g in (XVI), there exists a random field $X : \Delta \times M_2 \times \Omega \to M_2$ satisfying the following properties:

(i) For each $(v, \eta) \in M_2$, $(t_0, t) \in \Delta$, $X(t_0, t, (v, \eta), \cdot) = (x^{t_0,(v,\eta)}(t), x_t^{t_0,(v,\eta)})$ a.s., where $x^{t_0,(v,\eta)}$ is the unique solution of (XVI) with $x_{t_0}^{t_0,(v,\eta)} = (v, \eta)$.

(ii) For each $(t_0, t, \omega) \in \Delta \times \Omega$, the map

$$X(t_0, t, \cdot, \omega) : M_2 \to M_2$$

is C^∞.

(iii) For each $\omega \in \Omega$ and $(t_0, t) \in \Delta$ with $t > t_0 + r$, the map

$$X(t_0, t, \cdot, \omega) : M_2 \to M_2$$

carries bounded sets into relatively compact sets.

IV. ERGODIC THEORY OF REGULAR LINEAR SFDE's

Geilo, Norway

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14:00-14:50

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IV. ERGODIC THEORY OF LINEAR SFDE's

1. Plan

Use state space M_2. For regular linear sfde's (VIII), (IX), consider the following themes:

I) Existence of a "perfect" cocycle on M_2 that is a modification of the trajectory field $(x(t), x_t) \in M_2$.

II) Existence of almost sure Lyapunov exponents

$$\lim_{t\to\infty} \frac{1}{t} \log \|(x(t), x_t)\|_{M_2}$$

The multiplicative ergodic theorem and *hyperbolicity* of the cocycle.

III) *The Stable Manifold Theorem*, (viz. "random saddles") for hyperbolic systems.

2. Regular Linear Systems. White Noise

Linear sfde's on $\mathbf{R}^d$ driven by m-dimensional Brownian motion $W := (W_1, \cdots, W_m)$, with smooth coefficients.

$$\left.\begin{aligned} dx(t) &= H(x(t-d_1), \cdots, x(t-d_N), x(t), x_t)dt \\ &\qquad + \sum_{i=1}^{m} g_i x(t)\, dW_i(t), \quad t > 0 \\ (x(0), x_0) &= (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d) \end{aligned}\right\} \qquad (VIII)$$

(VIII) is defined on

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbf{R}}, P)$ = canonical complete filtered Wiener space.

$\Omega :=$ space of all continuous paths $\omega : \mathbf{R} \to \mathbf{R}^m$, $\omega(0) = 0$, in Euclidean space $\mathbf{R}^m$, with compact open topology;

$\mathcal{F} :=$ completed Borel σ-field of Ω;

$\mathcal{F}_t :=$ completed sub-σ-field of $\mathcal{F}$ generated by the evaluations $\omega \to \omega(u)$, $u \le t$, $t \in \mathbf{R}$.

$P :=$ Wiener measure on Ω.

$dW_i(t) =$ Itô stochastic differentials.

Several finite delays $0 < d_1 < d_2 < \cdots < d_N \le r$ in drift term; *no delays in diffusion coefficient.*

$H : (\mathbf{R}^d)^{N+1} \times L^2([-r, 0], \mathbf{R}^d) \to \mathbf{R}^d$ is a fixed continuous linear map, g_i, $i = 1, 2, \ldots, m$, fixed (deterministic) $d \times d$-matrices.

Recall regularity theorem:

Theorem III.4.([Mo], Stochastics, 1990])

(VIII) is regular with respect to the state space $M_2 = \mathbf{R}^d \times \mathbf{L}^2([-r,0],\mathbf{R}^d)$. There is a measurable version $X : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ of the trajectory field $\{(x(t),x_t) : t \in \mathbf{R}^+,\ (x(0),x_0) = (v,\eta) \in M_2\}$ of (VIII) with the following properties:

(i) For each $(v,\eta) \in M_2$ and $t \in \mathbf{R}^+$, $X(t,(v,\eta),\cdot) = (x(t),x_t)$ a.s., is $\mathcal{F}_t$-measurable and belongs to $L^2(\Omega,M_2;P)$.

(ii) There exists $\Omega_0 \in \mathcal{F}$ of full measure such that, for all $\omega \in \Omega_0$, the map $X(\cdot,\cdot,\omega) : \mathbf{R}^+ \times M_2 \to M_2$ is continuous.

(iii) For each $t \in \mathbf{R}^+$ and every $\omega \in \Omega_0$, the map $X(t,\cdot,\omega) : M_2 \to M_2$ is continuous linear; for each $\omega \in \Omega_0$, the map $\mathbf{R}^+ \ni t \mapsto X(t,\cdot,\omega) \in L(M_2)$ is measurable and locally bounded in the uniform operator norm on $L(M_2)$. The map $[r,\infty) \ni t \mapsto X(t,\cdot,\omega) \in L(M_2)$ is continuous for all $\omega \in \Omega_0$.

(iv) For each $t \geq r$ and all $\omega \in \Omega_0$, the map

$$X(t,\cdot,\omega) : M_2 \to M_2$$

is compact.

Compactness of semi-flow for $t \geq r$ will be used below to define hyperbolicity for (VIII) and the associated exponential dichotomies.

Lyapunov Exponents. Hyperbolicity

Version X of the flow constructed in Theorem III.4 is a multiplicative $L(M_2)$-valued linear cocycle over the canonical Brownian shift $\theta : \mathbf{R} \times \Omega \to \Omega$ on Wiener space:

$$\theta(t, \omega)(u) := \omega(t + u) - \omega(t), \quad u, t \in \mathbf{R}, \quad \omega \in \Omega.$$

Indeed we have

Theorem IV.1([M], 1990)

There is an $\mathcal{F}$-measurable set $\hat{\Omega}$ of full P-measure such that $\theta(t, \cdot)(\hat{\Omega}) \subseteq \hat{\Omega}$ for all $t \geq 0$ and

$$X(t_2, \cdot, \theta(t_1, \omega)) \circ X(t_1, \cdot, \omega) = X(t_1 + t_2, \cdot, \omega)$$

for all $\omega \in \hat{\Omega}$ and t_1, $t_2 \geq 0$.

The Cocycle Property

Proof of Theorem IV.1. (Sketch)

For simplicity consider the case of a single delay d_1; i.e. $N = 1$.

First step.

Approximate the Brownian motion W in (VIII) by smooth adapted processes $\{W^k\}_{k=1}^{\infty}$:

$$W^k(t) := k\int_{t-(1/k)}^{t} W(u)\,du - k\int_{-(1/k)}^{0} W(u)\,du, \quad t \geq 0,\ k \geq 1. \tag{1}$$

Exercise: Check that each W^k is a *helix* (i.e. has stationary increments):

$$W^k(t_1+t_2,\omega) - W^k(t_1,\omega) = W^k(t_2,\theta(t_1,\omega)), \quad t_1,t_2 \in \mathbf{R},\ \omega \in \Omega. \tag{2}$$

Let $X^k : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ be the stochastic (semi)flow of the random fde's:

$$\left.\begin{aligned} dx^k(t) &= H(x^k(t-d_1), x^k(t), x^k_t)dt \\ &\qquad + \sum_{i=1}^{m} g_i x(t)(W_i^k)'(t)\,dt - \frac{1}{2}\sum_{i=1}^{m} g_i^2 x^k(t)\,dt \quad t > 0 \\ (x^k(0), x^k_0) &= (v,\eta) \in M_2 := \mathbf{R}^d \times L^2([-r,0],\mathbf{R}^d) \end{aligned}\right\} \quad (VIII-k)$$

If $X : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ is the flow of (VIII) constructed in Theorem III.4, then

$$\lim_{k\to\infty} \sup_{0\leq t\leq T} \|X^k(t,\cdot,\omega) - X(t,\cdot,\omega)\|_{L(M_2)} = 0 \tag{3}$$

for every $0 < T < \infty$ and all ω in a Borel set $\hat{\Omega}$ of full Wiener measure which is invariant under $\theta(t,\cdot)$ for all $t \geq 0$ ([Mo], Stochastics, 1990). This convergence may be proved using the following stochastic variational method:

Let $\phi : \mathbf{R}^+ \times \Omega \to \mathbf{R}^{d\times d}$ be the $d \times d$-matrix-valued solution of the linear Itô sode (without delay):

$$\left.\begin{aligned} d\phi(t) &= \sum_{i=1}^{m} g_i \phi(t)\, dW_i(t) \qquad && t > 0 \\ \phi(0,\omega) &= I \in \mathbf{R}^{d\times d} && \text{a.a. } \omega \end{aligned}\right\} \tag{4}$$

Denote by $\phi^k : \mathbf{R}^+ \times \Omega \to \mathbf{R}^{d\times d}$, $k \geq 1$, the $d \times d$-matrix solution of the random family of linear ode's:

$$\left.\begin{aligned} d\phi^k(t) &= \sum_{i=1}^{m} g_i \phi^k(t)(W_i^k)'(t) - \frac{1}{2}\sum_{i=1}^{m} g_i^2 \phi^k(t)\, dt \qquad && t > 0 \\ \phi^k(0,\cdot) &= I \in \mathbf{R}^{d\times d}. && \end{aligned}\right\} \tag{4$'$}$$

Let $\hat{\Omega}$ be the sure event of all $\omega \in \Omega$ such that

$$\phi(t,\omega) := \lim_{k\to\infty} \phi^k(t,\omega) \tag{5}$$

exists uniformly for t in compact subsets of $\mathbf{R}^+$. Each ϕ^k is an $\mathbf{R}^{d\times d}$-valued *cocycle over* θ, viz.

$$\phi^k(t_1 + t_2, \omega) = \phi^k(t_2, \theta(t_1,\omega))\phi^k(t_1,\omega) \tag{6}$$

for all $t_1, t_2 \in \mathbf{R}^+$ and $\omega \in \Omega$. From the definition of $\hat{\Omega}$ and passing to the limit in (6) as $k \to \infty$, conclude that $\{\phi(t,\omega) : t > 0,\ \omega \in \Omega\}$, is an $\mathbf{R}^{d\times d}$-valued *perfect* cocycle over θ, viz.

(i) $P(\hat{\Omega}) = 1$;

(ii) $\theta(t,\cdot)(\hat{\Omega}) \subseteq \hat{\Omega}$ for all $t \geq 0$;

(iii) $\phi(t_1+t_2,\omega) = \phi(t_2,\theta(t_1,\omega))\phi(t_1,\omega)$ for all

$t_1, t_2 \in \mathbf{R}^+$ and *every* $\omega \in \hat{\Omega}$;

(iv) $\phi(\cdot,\omega)$ is continuous for every $\omega \in \hat{\Omega}$.

Alternatively use the perfection theorem in ([M-S], AIHP, 1996, Theorem 3.1, p. 79-82) for crude cocycles with values in a metrizable second countable topological group. Observe that $\phi(t,\omega) \in GL(\mathbf{R}^d)$.

Define $\hat{H} : \mathbf{R}^+ \times \mathbf{R}^d \times M_2 \times \Omega \to \mathbf{R}^d$ by

$$\begin{aligned} &\hat{H}(t, v_1, v, \eta, \omega) \\ &\quad := \phi(t,\omega)^{-1}[H(\phi_t(\cdot,\omega)(-d_1, v_1), \phi(t,\omega)(v), \phi_t(\cdot,\omega)\circ(id_J,\eta))] \end{aligned} \tag{7}$$

for $\omega \in \Omega, t \geq 0,\ v, v_1 \in \mathbf{R}^d,\ \eta \in L^2([-r,0],\mathbf{R}^d)$, where

$$\phi_t(\cdot,\omega)(s,v) = \begin{cases} \phi(t+s,\omega)(v) & t+s \geq 0 \\ \\ v & -r \leq t+s < 0 \end{cases}$$

and

$$(id_J,\eta)(s) = (s,\eta(s)), \qquad s \in J.$$

Define $\hat{H}^k : \mathbf{R}^+ \times \mathbf{R}^d \times M_2 \times \Omega \to \mathbf{R}^d$ by a relation similar to (7) with ϕ replaced by ϕ^k. Then the random fde's

$$\left.\begin{aligned} y'(t) &= \hat{H}(t, y(t-d_1), y(t), y_t, \omega) \qquad t > 0 \\ (y(0), y_0) &= (v,\eta) \in M_2 \end{aligned}\right\} \tag{8}$$

$$\left.\begin{array}{r}y^{k\prime}(t)=\hat{H}^k(t,y^k(t-d_1),y^k(t),y^k_t,\omega)\qquad t>0\\ (y^k(0),y^k_0)=(v,\eta)\in M_2\end{array}\right\}\tag{9}$$

have unique *non-explosive* solutions

$$y,\, y^k:[-r,\infty)\times\Omega\to\mathbf{R}^d$$

([Mo], Stochastics, 1990, pp. 93-98). Itô's formula implies that

$$X(t,v,\eta,\omega)=(\phi(t,\omega)(y(t,\omega)),\phi_t(\cdot,\omega)\circ(id_J,y_t))\tag{10}$$

The chain rule gives a similar relation for X^k with ϕ replaced by ϕ^k (*Exercise*; [Mo], Stochastics, 1990, pp. 96-97).

Get the convergence

$$\lim_{k\to\infty}|\hat{H}^k(t,v_1,v,\eta,\omega)-\hat{H}(t,v_1,v,\eta,\omega)|=0\tag{11}$$

uniformly for (t,v_1,v,η) in bounded sets of $\mathbf{R}^+\times\mathbf{R}^d\times M_2$. Use Gronwall's lemma and (11) to deduce (3).

Second step.

Fix $\omega\in\hat{\Omega}$ and use uniqueness of solutions to the approximating equation (VIII-k) and the helix property (2) of W^k to obtain the cocycle property for (X^k,θ):

$$X^k(t_2,\cdot,\theta(t_1,\omega))\circ X^k(t_1,\cdot,\omega)=X^k(t_1+t_2,\cdot,\omega)$$

for all $\omega\in\hat{\Omega}$ and $t_1,\ t_2\geq 0,\ k\geq 1$.

Third step.

Pass to limit as $k \to \infty$ in the above identity and use the convergence (3) *in operator norm* to get the perfect cocycle property for X.

$\square$

The a.s. Lyapunov exponents

$$\lim_{t\to\infty} \frac{1}{t} \log \|X(t,(v(\omega),\eta(\omega)),\omega)\|_{M_2},$$

(for a.a. $\omega \in \Omega$, $(v,\eta) \in L^2(\Omega, M_2)$) of the system (VIII) are characterized by the following "spectral theorem". Each $\theta(t,\cdot)$ is ergodic and preserves Wiener measure P. The proof of Theorem IV.2 below uses compactness of $X(t,\cdot,\omega) : M_2 \to M_2$, $t \geq r$, together with an infinite-dimensional version of Oseledec's multiplicative ergodic theorem due to Ruelle (1982).

Theorem IV.2. ([Mo], Stochastics, 1990)

Let $X : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ be the flow of (VIII) given in Theorem III.4. Then there exist

(a) an $\mathcal{F}$-measurable set $\Omega^* \subseteq \Omega$ such that $P(\Omega^*) = 1$ and $\theta(t,\cdot)(\Omega^*) \subseteq \Omega^*$ for all $t \geq 0$,

(b) a fixed (non-random) sequence of real numbers $\{\lambda_i\}_{i=1}^\infty$, and

(c) a random family $\{E_i(\omega) : i \geq 1, \omega \in \Omega^*\}$ of (closed) finite-codimensional subspaces of M_2, with the following properties:

(i) If the **Lyapunov spectrum** *$\{\lambda_i\}_{i=1}^\infty$ is infinite, then $\lambda_{i+1} < \lambda_i$ for all $i \geq 1$ and $\lim\limits_{i\to\infty} \lambda_i = -\infty$; otherwise there is a fixed (non-random) integer $N \geq 1$ such that $\lambda_N = -\infty < \lambda_{N-1} < \cdots < \lambda_2 < \lambda_1$;*

(ii) each map $\omega \mapsto E_i(\omega)$, $i \geq 1$, is $\mathcal{F}$-measurable into the Grassmannian of M_2;

(iii) $E_{i+1}(\omega) \subset E_i(\omega) \subset \cdots \subset E_2(\omega) \subset E_1(\omega) = M_2$, $i \geq 1$, $\omega \in \Omega^*$;

(iv) for each $i \geq 1$, *codim* $E_i(\omega)$ *is fixed independently of* $\omega \in \Omega^*$;

(v) for each $\omega \in \Omega^*$ *and* $(v, \eta) \in E_i(\omega) \backslash E_{i+1}(\omega)$,

$$\lim_{t \to \infty} \frac{1}{t} \log \|X(t, (v, \eta), \omega)\|_{M_2} = \lambda_i, \; i \geq 1;$$

(vi) **Top Exponent***:*

$$\lambda_1 = \lim_{t \to \infty} \frac{1}{t} \log \|X(t, \cdot, \omega)\|_{L(M_2)} \quad \text{for all } \omega \in \Omega^*;$$

(vii) **Invariance***:*

$$X(t, \cdot, \omega)(E_i(\omega)) \subseteq E_i(\theta(t, \omega))$$

for all $\omega \in \Omega^*, \; t \geq 0, \; i \geq 1.$

Proof of Theorem IV.2 is based on Ruelle's discrete version of Oseledec's multiplicative ergodic theorem in Hilbert space ([Ru], Ann. of Math. 1982, Theorem (1.1), p. 248 and Corollary (2.2), p. 253):

Theorem IV.3 ([Ru], 1982)

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\tau : \Omega \to \Omega$ a P-preserving transformation. Assume that H is a separable Hilbert space and $T : \Omega \to L(H)$ a measurable map (w.r.t. the Borel field on the space of all bounded linear operators $L(H)$). Suppose that $T(\omega)$ is compact for almost all $\omega \in \Omega$, and $E \log^+ \|T(\cdot)\| < \infty$. Define the family of linear operators $\{T^n(\omega) : \omega \in \Omega,\ n \geq 1\}$ by

$$T^n(\omega) := T(\tau^{n-1}(\omega)) \circ \cdots T(\tau(\omega)) \circ T(\omega)$$

for $\omega \in \Omega,\ n \geq 1$.

Then there is a set $\Omega_0 \in \mathcal{F}$ of full P-measure such that $\tau(\Omega_0) \subseteq \Omega_0$, and for each $\omega \in \Omega_0$, the limit

$$\lim_{n \to \infty} [T^n(\omega)^* \circ T^n(\omega)]^{1/(2n)} := \Lambda(\omega)$$

exists in the uniform operator norm and is a positive compact self-adjoint operator on H. Furthermore each $\Lambda(\omega)$ has a discrete spectrum

$$e^{\mu_1(\omega)} > e^{\mu_2(\omega)} > e^{\mu_3(\omega)} > e^{\mu_4(\omega)} > \cdots$$

where the μ_i's are distinct. If $\{\mu_i\}_{i=1}^{\infty}$ is infinite, then $\mu_i \downarrow -\infty$; otherwise they terminate at $\mu_{N(\omega)} = -\infty$. If $\mu_i(\omega) > -\infty$, then $e^{\mu_i(\omega)}$ has finite multiplicity $m_i(\omega)$ and finite-dimensional eigen-space $F_i(\omega)$, with $m_i(\omega) := \mathrm{dim} F_i(\omega)$. Define

$$E_1(\omega) := M_2, \quad E_i(\omega) := \left[\oplus_{j=1}^{i-1} F_j(\omega)\right]^{\perp}, \quad E_\infty(\omega) := \mathrm{ker}\, \Lambda(\omega).$$

Then

$$E_\infty(\omega) \subset \cdots \subset E_{i+1}(\omega) \subset E_i(\omega) \cdots \subset E_2(\omega) \subset E_1(\omega) = H$$

and

$$\lim_{n\to\infty} \frac{1}{n} \log \|T^n(\omega)x\|_H = \begin{cases} \mu_i(\omega), & \text{if} \quad x \in E_i(\omega)\backslash E_{i+1}(\omega) \\ -\infty & \text{if} \quad x \in \text{ker } \Lambda(\omega). \end{cases}$$

Proof.

[Ru], Ann. of Math., 1982, pp. 248-254. □

The following "perfect" version of Kingman's subadditive ergodic theorem is also used to construct the shift invariant set Ω^* appearing in Theorem IV.2 above.

Theorem IV.4([M, 1990])("Perfect" Subadditive Ergodic Theorem)

Let $f : \mathbf{R}^+ \times \Omega \to \mathbf{R} \cup \{-\infty\}$ be a measurable process on the complete probability space $(\Omega, \mathcal{F}, P)$ such that

(i) $E \sup_{0 \le u \le 1} f^+(u, \cdot) < \infty$, $E \sup_{0 \le u \le 1} f^+(1-u, \theta(u, \cdot)) < \infty$;

(ii) $f(t_1 + t_2, \omega) \le f(t_1, \omega) + f(t_2, \theta(t_1, \omega))$ *for all* $t_1, t_2 \ge 0$ *and* **every** $\omega \in \Omega$.

Then there exist a set $\hat{\hat{\Omega}} \in \mathcal{F}$ *and a measurable* $\tilde{f} : \Omega \to \mathbf{R} \cup \{-\infty\}$ *with the properties:*

(a) $P(\hat{\hat{\Omega}}) = 1,\ \theta(t, \cdot)(\hat{\hat{\Omega}}) \subseteq \hat{\hat{\Omega}}$ *for all* $t \ge 0$;

(b) $\tilde{f}(\omega) = \tilde{f}(\theta(t, \omega))$ *for all* $\omega \in \hat{\hat{\Omega}}$ *and all* $t \ge 0$;

(c) $\tilde{f}^+ \in \mathbf{L}^1(\Omega, \mathbf{R}; P)$;

(d) $\lim_{t \to \infty} (1/t) f(t, \omega) = \tilde{f}(\omega)$ *for every* $\omega \in \hat{\hat{\Omega}}$.

If θ *is ergodic, then there exist* $f^* \in \mathbf{R} \cup \{-\infty\}$ *and* $\tilde{\tilde{\Omega}} \in \mathcal{F}$ *such that*

(a)′ $P(\tilde{\tilde{\Omega}}) = 1, \theta(t, \cdot)(\tilde{\tilde{\Omega}}) \subseteq \tilde{\tilde{\Omega}}$, $t \ge 0$;

(b)′ $\tilde{f}(\omega) = f^* = \lim_{t \to \infty} (1/t) f(t, \omega)$ *for every* $\omega \in \tilde{\tilde{\Omega}}$.

Proof.

[Mo], Stochastics, 1990, Lemma 7, pp. 115–117. □

Proof of Theorem IV.2 is an application of Theorem IV.3. Requires Theorem IV.4 and the following sequence of lemmas.

Lemma 1

For each integer $k \geq 1$ and any $0 < a < \infty$,

$$E \sup_{0 \leq t \leq a} \|\phi(t,\omega)^{-1}\|^{2k} < \infty;$$

$$E \sup_{0 \leq t_1, t_2 \leq a} \|\phi(t_2, \theta(t_1, \cdot))\|^{2k} < \infty.$$

Proof.

Follows by standard sode estimates, the cocycle property for ϕ and Hölder's inequality. ([M], pp. 106-108). $\square$

The next lemma is a crucial estimate needed to apply Ruelle-Oseledec theorem (Theorem IV.3).

Lemma 2

$$E \sup_{0 \leq t_1, t_2 \leq r} \log^+ \|X(t_2, \cdot, \theta(t_1, \cdot))\|_{L(M_2)} < \infty.$$

Proof.

If $y(t,(v,\eta),\omega)$ is the solution of the fde (8), then using Gronwall's inequality, taking $E \sup_{0\le t_1,t_2\le r} \log^+ \sup_{\|(v,\eta)\|\le 1}$ and applying Lemma 1, gives

$$E \sup_{0\le t_1,t_2\le r} \log^+ \sup_{\|(v,\eta)\|\le 1} \|(y(t_2,(v,\eta),\theta(t_1,\cdot)), y_{t_2}(\cdot,(v,\eta),\theta(t_1,\cdot)))\|_{M_2} < \infty.$$

Conclusion of lemma now follows by replacing ω' with $\theta(t_1,\omega)$ in the formula

$$\begin{aligned} &X(t_2,(v,\eta),\omega') \\ &\quad = (\phi(t_2,\omega')(y(t_2,(v,\eta),\omega')), \phi_{t_2}(\cdot,\omega') \circ (id_J, y_{t_2}(\cdot,(v,\eta),\omega')) \end{aligned}$$

and Lemma 1. $\square$

The existence of the Lyapunov exponents is obtained by interpolating the discrete limit

$$\frac{1}{r} \lim_{k\to\infty} \frac{1}{k} \log \|X(kr,(v(\omega),\eta(\omega)),\omega)\|_{M_2}, \tag{12}$$

a.a. $\omega \in \Omega$, $(v,\eta) \in L^2(\Omega, M_2)$, between delay periods of length r. This requires the next two lemmas.

Lemma 3

Let $h : \Omega \to \mathbf{R}^+$ be $\mathcal{F}$-measurable and suppose $E \sup_{0 \le u \le r} h(\theta(u, \cdot)$ is finite. Then

$$\Omega_1 := \big(\lim_{t\to\infty} \frac{1}{t} h(\theta(t, \cdot) = 0\big)$$

is a sure event and $\theta(t, \cdot)(\Omega_1) \subseteq \Omega_1$ for all $t \ge 0$.

Proof.

Use interpolation between delay periods and the discrete ergodic theorem applied to the L^1 function

$$\hat{h} := \sup_{0 \le u \le r} h(\theta(u, \cdot).$$

([Mo], Stochastics, 1990, Lemma 5, pp. 111-113.) $\square$

Lemma 4

Suppose there is a sure event Ω_2 such that $\theta(t, \cdot)(\Omega_2) \subseteq \Omega_2$ for all $t \ge 0$, and the limit (12) exists (or equal to $-\infty$) for all $\omega \in \Omega_2$ and all $(v, \eta) \in M_2$. Then there is a sure event Ω_3 such that $\theta(t, \cdot)(\Omega_3) \subseteq \Omega_3$ and

$$\lim_{t\to\infty} \frac{1}{t} \log \|X(t, (v, \eta), \omega)\|_{M_2} = \frac{1}{r} \lim_{k\to\infty} \frac{1}{k} \log \|X(kr, (v, \eta), \omega)\|_{M_2}, \tag{13}$$

for all $\omega \in \Omega_3$ and all $(v, \eta) \in M_2$.

Proof:

Take $\Omega_3 := \hat{\Omega} \cap \Omega_1 \cap \Omega_2$. Use cocycle property for X, Lemma 2 and Lemma 3 to interpolate. ([Mo], Stochastics 1990, Lemma 6, pp. 113-114.) $\square$

Proof of Theorem IV.2. (Sketch)

Apply Ruelle-Oseledec Theorem (Theorem IV.3) with

$T(\omega) := X(r, \omega) \in L(M_2)$, compact linear for $\omega \in \hat{\Omega}$;

$\tau : \Omega \to \Omega; \quad \tau := \theta(r, \cdot)$.

Then cocycle property for X implies

$$\begin{aligned} X(kr, \omega, \cdot) &= T(\tau^{k-1}(\omega)) \circ T(\tau^{k-2}(\omega)) \circ \cdots \circ T(\tau(\omega)) \circ T(\omega) \\ &:= T^k(\omega) \end{aligned}$$

for all $\omega \in \hat{\Omega}$.

Lemma 2 implies

$$E \log^+ \|T(\cdot)\|_{L(M_2)} < \infty.$$

Theorem IV.3 gives a random family of compact self-adjoint positive linear operators $\{\Lambda(\omega) : \omega \in \Omega_4\}$ such that

$$\lim_{n\to\infty} [T^n(\omega)^* \circ T^n(\omega)]^{1/(2n)} := \Lambda(\omega)$$

exists in the uniform operator norm and is a positive compact operator on M_2 for $\omega \in \Omega_4$, a (continuous) shift-invariant set of full measure. Furthermore each $\Lambda(\omega)$ has a discrete spectrum

$$e^{\mu_1(\omega)} > e^{\mu_2(\omega)} > e^{\mu_3(\omega)} > e^{\mu_4(\omega)} > \cdots$$

where the μ_i's are distinct, with no accumulation points except possibly $-\infty$. If $\{\mu_i\}_{i=1}^{\infty}$ is infinite, then $\mu_i \downarrow -\infty$; otherwise they terminate at

$\mu_{N(\omega)} = -\infty$. If $\mu_i(\omega) > -\infty$, then $e^{\mu_i(\omega)}$ has finite multiplicity $m_i(\omega)$ and finite-dimensional eigen-space $F_i(\omega)$, with $m_i(\omega) := dim F_i(\omega)$. Define

$$E_1(\omega) := M_2, \quad E_i(\omega) := \left[\oplus_{j=1}^{i-1} F_j(\omega)\right]^{\perp}, \quad E_\infty(\omega) := \text{ker } \Lambda(\omega).$$

Then

$$E_\infty(\omega) \subset \cdots \subset E_{i+1}(\omega) \subset E_i(\omega) \cdots \subset E_2(\omega) \subset E_1(\omega) = M_2.$$

Note that $\operatorname{codim} E_i(\omega) = \sum_{j=1}^{i-1} m_j(\omega) < \infty$. Also

$$\lim_{k\to\infty} \frac{1}{k} \log \|X(kr,(v,\eta),\omega)\|_{M_2} = \begin{cases} \mu_i(\omega), & \text{if } (v,\eta) \in E_i(\omega)\backslash E_{i+1}(\omega) \\ -\infty & \text{if } (v,\eta) \in \text{ker } \Lambda(\omega). \end{cases}$$

The functions

$$\omega \mapsto \mu_i(\omega), \quad \omega \mapsto m_i(\omega), \quad \omega \mapsto N(\omega)$$

are invariant under the ergodic shift $\theta(r,\cdot)$. Hence they take the fixed values μ_i, m_i, N almost surely, respectively.

Lemma 4 gives a continuous-shift-invariant sure event $\Omega^* \subseteq \Omega_4$ such that

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{t} \log \|X(t,(v,\eta),\omega)\|_{M_2} &= \frac{1}{r} \lim_{k\to\infty} \frac{1}{k} \log \|X(kr,(v,\eta),\omega)\|_{M_2} \\ &= \frac{\mu_i}{r} =: \lambda_i, \end{aligned}$$

for $(v,\eta) \in E_i(\omega)\backslash E_{i+1}(\omega)$, $\omega \in \Omega^*, i \geq 1$.

$\{\lambda_i := \frac{\mu_i}{r} : i \geq 1\}$ is the *Lyapunov spectrum* of (VIII).

Since Lyapunov spectrum is discrete with no finite accumulation points, then $\{\lambda_i : \lambda_i > \lambda\}$ is finite for all $\lambda \in \mathbf{R}$.

To prove invariance of the Oseledec space $E_i(\omega)$ under the cocycle (X, θ) use the random field

$$\lambda((v,\eta),\omega) := \lim_{t\to\infty} \frac{1}{t} \log \|X(t,(v,\eta),\omega)\|_{M_2} \qquad (v,\eta) \in M_2, \quad \omega \in \Omega^*$$

and the relations

$$E_i(\omega) := \{(v,\eta) \in M_2 : \lambda((v,\eta),\omega) \leq \lambda_i\},$$

$$\lambda(X(t,(v,\eta),\omega),\theta(t,\omega)) = \lambda((v,\eta),\omega), \quad \omega \in \Omega^*, \ t \geq 0$$

([Mo], Stochastics 1990, p. 122). □

The non-random nature of the Lyapunov exponents $\{\lambda_i\}_{i=1}^{\infty}$ of (VIII) is a consequence of the fact the θ is ergodic. (VIII) is said to be *hyperbolic* if $\lambda_i \neq 0$ for all $i \geq 1$. When (VIII) is hyperbolic the flow satisfies a *stochastic saddle-point property* (or exponential dichotomy) (cf. the deterministic case with $E = C([-r,0],\mathbf{R}^d)$, $g_i \equiv 0$, $i = 1, \ldots, m$, in Hale [H], Theorem 4.1, p. 181).

Theorem IV.5 *(Random Saddles)*([Mo], Stochastics, 1990)

Suppose the sfde (VIII) is hyperbolic. Then there exist

(a) a set $\tilde{\Omega}^* \in \mathcal{F}$ *such that* $P(\tilde{\Omega}^*) = 1$, *and* $\theta(t,\cdot)(\tilde{\Omega}^*) = \tilde{\Omega}^*$ *for all* $t \in \mathbf{R}$,

and

(b) a measurable splitting

$$M_2 = \mathcal{U}(\omega) \oplus \mathcal{S}(\omega), \qquad \omega \in \tilde{\Omega}^*,$$

with the following properties:

(i) $\mathcal{U}(\omega)$, $\mathcal{S}(\omega)$, $\omega \in \tilde{\Omega}^*$, *are closed linear subspaces of* M_2, $\dim \mathcal{U}(\omega)$ *is finite and fixed independently of* $\omega \in \tilde{\Omega}^*$.

(ii) The maps $\omega \mapsto \mathcal{U}(\omega)$, $\omega \mapsto \mathcal{S}(\omega)$ *are* $\mathcal{F}$*-measurable into the Grassmannian of* M_2.

(iii) For each $\omega \in \tilde{\Omega}^*$ *and* $(v,\eta) \in \mathcal{U}(\omega)$ *there exists* $\tau_1 = \tau_1(v,\eta,\omega) > 0$ *and a positive* δ_1, *independent of* (v,η,ω) *such that*

$$\|X(t,(v,\eta),\omega)\|_{M_2} \geq \|(v,\eta)\|_{M_2} e^{\delta_1 t}, \quad t \geq \tau_1.$$

(iv) For each $\omega \in \tilde{\Omega}^*$ and $(v, \eta) \in \mathcal{S}(\omega)$ there exists $\tau_2 = \tau_2(v, \eta, \omega) > 0$ and a positive δ_2, independent of (v, η, ω) such that

$$\|X(t, (v, \eta), \omega)\|_{M_2} \leq \|(v, \eta)\|_{M_2} e^{-\delta_2 t}, \quad t \geq \tau_2.$$

(v) For each $t \geq 0$ and $\omega \in \tilde{\Omega}^*$,

$$X(t, \omega, \cdot)(\mathcal{U}(\omega)) = \mathcal{U}(\theta(t, \omega)),$$

$$X(t, \omega, \cdot)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t, \omega)).$$

In particular, the restriction

$$X(t, \omega, \cdot) \,|\, \mathcal{U}(\omega) : \mathcal{U}(\omega) \to \mathcal{U}(\theta(t, \omega))$$

is a linear homeomorphism onto.

Proof.

[Mo], Stochastics, 1990, Corollary 2, pp. 127-130. □

The Stable Manifold Theorem

5. Regular Linear Systems. Helix Noise

$$\left.\begin{aligned} dx(t) = \Big\{ \int_{[-r,0]} \nu(t)(ds)\, x(t+s) \Big\}\, dt + dN(t) \int_{-r}^{0} K(t)(s)\, x(t+s)\, ds \\ + dL(t)\, x(t-), \quad t > 0 \\ (x(0), x_0) = (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r,0], \mathbf{R}^d) \end{aligned}\right\} \qquad (IX)$$

Linear systems driven by helix semimartingale noise, and memory driven by a measure-valued process ν on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbf{R}}, P)$.

Hypotheses (C)

(i) The processes ν, K are stationary ergodic in the sense that there is a measurable ergodic P-preserving flow $\theta : \mathbf{R} \times \Omega \to \Omega$ such that for each $t \in \mathbf{R}$, $\mathcal{F}_t = \theta(t,\cdot)^{-1}(\mathcal{F}_0)$ and

$$\nu(t,\omega) = \nu(0, \theta(t,\omega)), \quad t \in \mathbf{R},\ \omega \in \Omega$$

$$K(t,\omega) = K(0, \theta(t,\omega)), \quad t \in \mathbf{R},\ \omega \in \Omega.$$

(ii) $L = M + V$, M continuos local martingale, V B.V. process. The processes N, L, M have jointly stationary ergodic increments:

$$N(t+h,\omega) - N(t,\omega) = N(h, \theta(t,\omega)),$$

$$L(t+h,\omega) - L(t,\omega) = L(h, \theta(t,\omega)),$$

$$M(t+h,\omega) - M(t,\omega) = M(h, \theta(t,\omega)),$$

for $t \in \mathbf{R}$, $\omega \in \Omega$.

Semimartingales satisfying Hypothesis (C)(ii) were studied by de Sam Lazaro and P.A. Meyer ([S-M], 1971, 1976), Çinlar, Jacod, Protter and Sharpe [CJPS], Protter [P], 1986.

Equation (IX) is regular w.r.t. M_2 with a measurable flow $X : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$. This flow satisfies Theorems III.4 and the cocycle property. This is achieved via a construction in ([M-S], AIHP, 1996) based on the following consequence of Hypothesis (C)(ii):

Theorem IV.6 ([Mo], Survey paper, 1992, [M-S], AIHP, 1996)

Suppose M satisfies Hypothesis (C)(ii). Then there is an $(\mathcal{F}_t)_{t\geq 0}$-adapted version $\phi : \mathbf{R}^+ \times \Omega \to \mathbf{R}^{d\times d}$ of the solution to the matrix equation

$$\left.\begin{aligned} d\phi(t) &= dM(t)\phi(t) \quad t>0 \\ \phi(0) &= I \in \mathbf{R}^{d\times d} \end{aligned}\right\} \qquad (X)$$

and a set $\Omega_1 \in \mathcal{F}$ such that

(i) $P(\Omega_1) = 1$*;*

(ii) $\theta(t,\cdot)(\Omega_1) \subseteq \Omega_1$ *for all* $t \geq 0$*;*

(iii) $\phi(t_1 + t_2, \omega) = \phi(t_2, \theta(t_1, \omega))\phi(t_1, \omega)$ *for all* $t_1, t_2 \in \mathbf{R}^+$ *and every* $\omega \in \Omega_1$*;*

(iv) $\phi(\cdot, \omega)$ *is continuous for every* $\omega \in \Omega_1$*.*

A proof of Theorem IV.6 is given in ([Mo], Survey, 1992; [M-S], AIHP, 1996): either by a double-approximation argument or via perfection techniques.

The existence of a discrete non-random Lyapunov spectrum $\{\lambda_i\}_{i=1}^{\infty}$ for the sfde (IX) is proved via Ruelle-Oseledec multiplicative ergodic theorem which requires the integrability property (Lemma 2):

$$E \sup_{0\leq t_1, t_2 \leq r} \log^+ \|X(t_1, \theta(t_2, \cdot), \cdot)\|_{L(M_2)} < \infty.$$

The above integrability property is established under the following set of hypotheses on ν, K, N, L:

Hypotheses (I)

(i)

$$\sup_{-r\le s\le 2r}\left|\frac{d\bar{\bar{\nu}}(\cdot)(s)}{ds}\right|^2, \qquad \sup_{0\le t\le 2r, -r\le s\le 0}\|K(t,\cdot)(s)\|^3,$$
$$\sup_{0\le t\le 2r, -r\le s\le 0}\|\frac{\partial}{\partial t}K(t,\cdot)(s)\|^3, \qquad \sup_{0\le t\le 2r, -r\le s\le 0}\|\frac{\partial}{\partial s}K(t,\cdot)(s)\|^3,$$
$$\{|V|(2r,\cdot)\}^4,$$

are all integrable, where

$$\bar{\bar{\nu}}(\omega)(A) := \int_0^\infty |\nu(t,\omega)|\{(A-t)\cap[-r,0]\}\,dt, \qquad A\in Borel[-r,\infty)$$

has a locally (essentially) bounded density $\dfrac{d\bar{\bar{\nu}}(\cdot)(s)}{ds}$; and $|V| =$ total variation of V w.r.t. the Euclidean norm $\|\cdot\|$ on $\mathbf{R}^{d\times d}$.

(ii) Let $N = N^0 + V^0$ where the local $(\mathcal{F}_t)_{t\ge 0}$-martingale $N^0 = (N^0_{ij})^d_{i,j=1}$ and the bounded variation process

$V^0 = (V^0_{ij})^d_{i,j=1}$ are such that

$$\{[N^0_{ij}](2r,\cdot)\}^2, \quad \{|V^0_{ij}|(2r,\cdot)\}^4, \quad i,j = 1,2,\ldots,d$$

are integrable.

$|V^0_{ij}|(2r,\cdot) =$ total variation of V^0_{ij} over $[0,2r]$.

(iii) $[M_{ij}](1)\in L^\infty(\Omega,\mathbf{R})$, $\quad i,j = 1,2,\ldots,d.$

The integrability property of the cocycle (X,θ) is a consequence of

$$E\log^+\sup_{0\le t_1,t_2\le r,\ \|(v,\eta)\|\le 1}|x(t_1,(v,\eta),\theta(t_2,\cdot))| < \infty.$$

Proof of latter property uses lengthy argument based on establishing the existence of suitable higher order moments for the coefficients of an associated random integral equation. (See Lemmas (5.1)-(5.5) in [M-S],I, AIHP, 1996.)

Since θ is ergodic, the multiplicative ergodic theorem (Theorem IV.3, Ruelle) now gives a fixed discrete set of Lyapunov exponents

Theorem IV.7 ([Mo], Survey, 1992; [M-S], AIHP, 1996)

Under Hypotheses (C) & (I), the statements of Theorems IV.2 and IV.5 hold true for the linear sfde (IX).

Note that the Lyapunov spectrum of (IX) does not change if one uses the state space $D([-r,0],\mathbf{R}^d)$ with the supremum norm $\|\cdot\|_\infty$ ([M-S], AIHP 1996).

V. STABILITY

EXAMPLES AND CASE STUDIES

Geilo, Norway

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V. STABILITY. EXAMPLES AND CASE STUDIES

1. Plan.

I) Estimates on the "maximal exponential growth rate" for the singular noisy feedback loop. Use of Lyapunov functionals.

II) Examples and case studies of linear sfde's: Existence of the stochastic semiflow and its Lyapunov spectrum.

III) Study almost sure asymptotic stability via upper bounds on the top Lyapunov exponent λ_1.

IV) Lyapunov spectrum for sdde's with Poisson noise.

Lyapunov exponents for linear sode's (without memory): studied by many authors: e.g. Arnold, Kliemann and Oeljeklaus, 1989, Arnold, Oeljeklaus and Pardoux, 1986, Baxendale, 1985, Pardoux and Wihstutz[PW1], 1988, Pinsky and Wihstutz [PW2], 1988, and the references therein.

Asymptotic stability of sfde's: treated in Kushner [K], JDE, 1968, Mizel and Trutzer [MT],1984, Mohammed [M1]-[M4], 1984, 1986, 1990, 1992, Mohammed and Scheutzow [MS], 1996, Scheutzow [S], 1988, Kolmanovskii and Nosov [KN], 1986. Mao ([Ma], 1994, Chapter 5) gives several results concerning top exponential growth rate for sdde's driven by C-valued semimartingales. Assumes that second-order characteristics of the driving semimartingales are *time-dependent* and *decay to zero exponentially fast in time, uniformly in the space variable.*

2. Noisy Feedback Loop Revisited Once More!

Noisy feedback loop is modelled by the one-dimensional linear sdde

$$\left.\begin{aligned} dx(t) &= \sigma x(t-r))\,dW(t), \qquad t>0 \\ (x(0),x_0) &= (v,\eta)\in M_2 := \mathbf{R}\times L^2([-r,0],\mathbf{R}), \end{aligned}\right\} \tag{I}$$

driven by a Wiener process W with a *positive delay* r.

(I) is singular with respect to M_2 (Theorem III.3).

Consider the more general one-dimensional linear sfde:

$$\left.\begin{aligned} dx(t) &= \int_{-r}^{0} x(t+s)d\nu(s)\,dW(t), \qquad t>0 \\ (x(0),x_0) &\in M_2 := \mathbf{R}\times L^2([-r,0],\mathbf{R}) \end{aligned}\right\} \tag{II'}$$

where W is a Wiener process and ν is a fixed finite real-valued Borel measure on $[-r,0]$.

(II') is regular if ν has a C^1 (or even L_1^2) density with respect to Lebesgue measure on $[-r,0]$ ([M-S], I, 1996). If ν satisfies Theorem III.3, then (II') is singular.

In the singular case, there is no stochastic flow (Theorem III.3) and we do not know whether a (discrete) set of Lyapunov exponents

$$\lambda((v,\eta),\cdot) := \lim_{t\to\infty}\frac{1}{t}\log\|(x(t,(v,\eta)),x_t(\cdot,(v,\eta)))\|_{M_2}, \qquad (v,\eta)\in M_2$$

exists. Existence of Lyapunov exponents for singular equations is hard. But can still define the *maximal exponential growth rate*

$$\overline{\lambda}_1 := \sup_{(v,\eta)\in M_2}\ \limsup_{t\to\infty}\frac{1}{t}\log\|(x(t,(v,\eta)),x_t(\cdot,(v,\eta)))\|_{M_2}$$

for the trajectory random field $\{(x(t,(v,\eta)),x_t(\cdot,(v,\eta))) : t\ge 0, (v,\eta)\in M_2\}$. $\overline{\lambda}_1$ may depend on $\omega\in\Omega$. But $\overline{\lambda}_1=\lambda_1$ in the regular case.

Inspite of the *extremely erratic dependence on the initial paths* of solutions of (I), it is shown in Theorem V.1 that for small noise variance, *uniform almost sure global asymptotic stability* still persists. For small σ, $\overline{\lambda}_1 \leq -\sigma^2/2 + o(\sigma^2)$ uniformly in the initial path (Theorem V.1, and Remark (iii)). For large $|\sigma|$ and $\nu = \delta_{-r}$,

$$\frac{1}{2r} \log |\sigma| + o(\log |\sigma|) \leq \overline{\lambda}_1 \leq \frac{1}{r} \log |\sigma|$$

([M-S], II, 1996, Remark (ii) after proof ofTheorem 2.3). This result is in sharp contrast with the non-delay case ($r = 0$), where $\lambda_1 = -\sigma^2/2$ *for all* values of σ. Proofs of Theorems V.1, V.2 involve very delicate constructions of new types of Lyapunov functionals on the underlying state space.

Theorem V.1.([M-S], II, 1996).

Let ν be a probability measure on $[-r, 0]$, $r > 0$, and consider the sfde

$$\left.\begin{aligned} dx(t) &= \sigma\left(\int_{[-r,0]} x(t+s)\, d\nu(s)\right) dW(t), \quad t \geq 0 \\ (x(0), x_0) &= (v, \eta) \in M_2 \end{aligned}\right\} \qquad (II')$$

with $\sigma \in \mathbf{R}$, $(v, \eta) \in M_2$, W standard Brownian motion, and $x(\cdot, (v, \eta))$ the solution of (II') through $(v, \eta) \in M_2$. Then there exists $\sigma_0 > 0$ and a continuous strictly negative nonrandom function $\phi : (-\sigma_0, \sigma_0) \to \mathbf{R}^-$ (independent of $(v, \eta) \in M_2$ and ν) such that

$$P\left(\limsup_{t\to\infty} \frac{1}{t} \log \|(x(t, (v, \eta)), x_t(\cdot, (v, \eta)))\|_{M_2} \leq \phi(\sigma)\right) = 1.$$

for all $(v, \eta) \in M_2$ and all $-\sigma_0 < \sigma < \sigma_0$.

Remark:

Theorem also holds for state space C with $\|\cdot\|_\infty$.

Proof of Theorem V.1. (Sketch)

Sufficient to consider (II') on $C \equiv C([-r,0],\mathbf{R})$, because C is continuously embedded in M_2. W.l.o.g., assume that $\sigma > 0$.

- Use Lyapunov functional $V : C \to \mathbf{R}^+$

$$V(\eta) := (R(\eta) \vee |\eta(0)|)^\alpha + \beta R(\eta)^\alpha, \quad \eta \in C.$$

  where $R(\eta) := \overline{\eta} - \underline{\eta}$, the diameter of the range of η, $\overline{\eta} := \sup_{-r \le s \le 0} \eta(s)$ and $\underline{\eta} := \inf_{-r \le s \le 0} \eta(s)$.

- Fix $0 < \alpha < 1$ and *arrange* for $\beta = \beta(\sigma)$ for sufficiently small σ such that

$$E(V({}^\eta x_r)) \le \delta(\sigma) V(\eta), \quad \eta \in C, \tag{1}$$

  and $\delta(\sigma) \in (0,1)$ is a continuous function of σ defined near 0. There is a positive $K = K(\alpha)$ (independent of η, ν) such that $\delta(\sigma) \sim (1 - K\sigma^2)$. Set

$$\phi(\sigma) := \frac{1}{\alpha} \log \delta(\sigma).$$

  Estimate (1) is hard ([M-S], II, 1996, pp. 12-18).

- $\{{}^\eta x_{nr}\}_{n=1}^\infty$ is a Markov process in C. So (1) implies that $\delta(\sigma)^{-n} V({}^\eta x_{nr}),\ n \ge 1$, is a non-negative $(\mathcal{F}_{nr})$ supermartingale.

- There exists $Z : \Omega \to [0,\infty)$ such that

$$\lim_{n \to \infty} \frac{V({}^\eta x_{nr})}{\delta(\sigma)^n} = Z \qquad \text{a.s.} \tag{2}$$

- Form of V and (2) imply

$$\begin{aligned}\overline{\lim_{t \to \infty}} \frac{1}{t} \log |x(t)| &\le \overline{\lim_{n \to \infty}} \frac{1}{nr} \log[|x(nr)| + R(x_{nr})] \\ &= \frac{1}{\alpha} \overline{\lim_{n \to \infty}} \frac{1}{nr} \log V(x_{nr}) \le \frac{1}{\alpha} \log \delta(\sigma) = \phi(\sigma) < 0.\end{aligned}$$

- $\delta(\sigma)$, $\phi(\sigma)$ independent of η, ν. "Domain" of ϕ also independent of η, ν. $\square$

Remarks.

(i) Choice of σ_0 in Theorem V.1 depends on r. In (I) the scaling $t \mapsto t/r$ has the effect of replacing r by 1 and σ by $\sigma\sqrt{r}$. If $\overline{\lambda}_1(r,\sigma)$ is the maximal exponential growth rate of (I), then $\overline{\lambda}_1(r,\sigma) = \frac{1}{r}\overline{\lambda}_1(1,\sigma\sqrt{r})$ (*Exercise*). Hence σ_0 decreases (like $\frac{1}{\sqrt{r}}$) as r increases . Thus (for a fixed σ), a *small delay* r tends to *stabilize* equation (I). A *large delay* in (I) has a *destabilizing* effect (Theorem V.2 below).

(ii) Using a Lyapunov function(al) argument, Theorem V.2 below shows that for sufficiently large σ, the singular delay equation (I) is unstable. Result is in sharp contrast with the non-delay case $r = 0$, where

$$\lim_{t\to\infty} \frac{1}{t} \log|x(t)| = -\sigma^2/2 < 0$$

for *all* $\sigma \in \mathbf{R}$ (even when σ is large).

(iii) The growth rate function ϕ in Theorem V.1 satisfies

$$\phi(\sigma) = -\sigma^2/2 + o(\sigma^2)$$

as $\sigma \to 0^+$. Agrees with non-delay case $r = 0$. Above relation follows by modifying proof of Theorem V.1.

Theorem V.2.

Consider the equation

$$\left.\begin{aligned} dx(t) &= \sigma x(t-r)\,dW(t), \quad t>0 \\ (x(0),x_0) &= (v,\eta) \in M_2 := \mathbf{R} \times L^2([-r,0],\mathbf{R}), \end{aligned}\right\} \qquad (I)$$

driven by a standard Wiener process W with a positive delay r and $\sigma \in \mathbf{R}$. Then there exists a continuous function $\psi : (0,\infty) \to \mathbf{R}$ which is increasing to infinity such that

$$P\left(\liminf_{t\to\infty} \frac{1}{t} \log \|(x(t,(v,\eta)), x_t(\cdot,(v,\eta))\|_{M_2} \geq \psi(|\sigma|) \right) = 1,$$

for all $(v,\eta) \in M_2 \backslash \{0\}$ and all $\sigma \neq 0$. The function ψ is independent of the choice of $(v,\eta) \in M_2 \backslash \{0\}$.

Remarks.

(i) $\|\cdot\|_{M_2}$ can be replaced by the sup-norm on C.

(ii) Proof shows $\psi(\sigma) \sim \frac{1}{2}\log\sigma$ for large σ.

Proof of Theorem V.2.

Use the continuous Lyapunov functional

$$V : M_2 \backslash \{0\} \to [0, \infty)$$

$$V((v, \eta)) := \left(v^2 + |\sigma| \int_{-1}^{0} \eta^2(s)\, ds \right)^{-1/4}$$

[M-S], Part II, 1996, pp. 20-24. $\square$

3. Regular one-dimensional linear sfde's

To outline a general scheme for obtaining estimates on the top Lyapunov exponent for a class of one-dimensional regular linear sfde's. Then apply scheme to specific examples within the above class.

Scheme applies to multidimensional linear equations with multiple delays.

Note: Approach in ([Ku], JDE, 1968) uses Lyapunov functionals and yields strictly weaker estimates in all cases.

Consider the class of one-dimensional linear sfde's

$$\left. \begin{aligned} dx(t) = \Big\{ \nu_1 x(t) + \mu_1 x(t-r) + \int_{-r}^{0} x(t+s)\sigma_1(s)\, ds \Big\}\, dt \\ + \Big\{ \nu_2 x(t) + \int_{-r}^{0} x(t+s)\sigma_2(s)\, ds \Big\}\, dM(t), \end{aligned} \right\} \qquad (XVII)$$

where $r > 0, \sigma_1, \sigma_2 \in C^1([-r, 0], \mathbf{R})$, and M is a continuous helix local martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ with (stationary) ergodic increments. Ergodic theorem gives the a.s. deterministic limit $\beta := \lim\limits_{t \to \infty} \dfrac{\langle M \rangle(t)}{t}$. Assume that $\beta < \infty$ and $\langle M \rangle(1) \in L^\infty(\Omega, \mathbf{R})$.

Hence (XVII) is regular with respect to M_2 and has a sample-continuous stochastic semiflow $X : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ (Theorem III.5). The stochastic semiflow X has a fixed (non-random) Lyapunov spectrum (Theorem IV.7). Let λ_1 be its top exponent. We wish to develop an upper bound for λ_1. By the spectral theorem (Theorem IV.7, cf. Theorem IV.2), there is a shift-invariant set $\Omega^* \in \mathcal{F}$ of full P-measure and a measurable random field $\lambda : M_2 \times \Omega \to \mathbf{R} \cup \{-\infty\}$,

$$\lambda((v,\eta),\omega) := \lim_{t\to\infty} \frac{1}{t} \log \|X(t,(v,\eta),\omega)\|_{M_2}, \quad (v,\eta) \in M_2,\ \omega \in \Omega^*, \tag{1}$$

giving the Lyapunov spectrum of (XVII).

Introduce family of equivalent norms

$$\|(v,\eta)\|_\alpha := \left\{\alpha v^2 + \int_{-r}^0 \eta(s)^2\, ds\right\}^{1/2}, \quad (v,\eta) \in M_2, \quad \alpha > 0, \tag{2}$$

on M_2. Then

$$\lambda((v,\eta),\omega) = \lim_{t\to\infty} \frac{1}{t} \log \|X(t,(v,\eta),\omega)\|_\alpha, \quad (v,\eta) \in M_2,\ \omega \in \Omega^* \tag{3}$$

for all $\alpha > 0$; i.e. the Lyapunov spectrum of (XVII) with respect to $\|\cdot\|_\alpha$ is independent of $\alpha > 0$.

Let x be the solution of (XVII) starting at $(v,\eta) \in M_2$. Define

$$\rho_\alpha(t)^2 := \|X(t)\|_\alpha^2 = \alpha x(t)^2 + \int_{t-r}^t x(u)^2\, du, \quad t > 0, \quad \alpha > 0. \tag{4}$$

For each fixed $(v,\eta) \in M_2$, define the set $\Omega_0 \in \mathcal{F}$ by $\Omega_0 := \{\omega \in \Omega : \rho_\alpha(t,\omega) \neq 0 \text{ for all } t > 0\}$. If $P(\Omega_0) = 0$, then by uniqueness there is a random time τ_0 such that a.s. $X(t,(v,\eta),\cdot) = 0$ for all $t \geq \tau_0$. Hence $\lambda_1 = -\infty$. So suppose that $P(\Omega_0) > 0$. Itô's formula implies

$$\begin{aligned}\log \rho_\alpha(t) = \log \rho_\alpha(0) &+ \int_0^t Q_\alpha(a(u), b(u), I_1(u))\, du \\ &+ \int_0^t \tilde{Q}_\alpha(a(u), I_2(u))\, d\langle M\rangle(u) + \int_0^t R_\alpha(a(u), I_2(u))\, dM(u),\end{aligned} \tag{5}$$

for $t > 0$, a.s. on Ω_0, where

$$\left.\begin{aligned} Q_\alpha(z_1, z_2, z_3) &:= \nu_1 z_1^2 + \sqrt{\alpha}\, \mu_1 z_1 z_2 + \sqrt{\alpha}\, z_1 z_3 + \tfrac{1}{2}\frac{z_1^2}{\alpha} - \tfrac{1}{2} z_2^2 \\ \tilde{Q}_\alpha(z_1, z_3') &:= \alpha(\tfrac{1}{2} - z_1^2)\left(\frac{\nu_2}{\sqrt{\alpha}}\, z_1 + z_3'\right)^2 \\ R_\alpha(z_1, z_3') &:= \nu_2 z_1^2 + \sqrt{\alpha} z_1 z_3', \qquad \|\sigma_i\|_2 := \left\{\int_{-r}^0 \sigma_i(s)^2 ds\right\}^{1/2}, \end{aligned}\right\} \tag{6}$$

$i = 1, 2$, and

$$a(t) := \frac{\sqrt{\alpha} x(t)}{\rho_\alpha(t)}, \quad b(t) := \frac{x(t-r)}{\rho_\alpha(t)}, \quad I_i(t) := \frac{\int_{-r}^0 x(t+s)\sigma_i(s)\, ds}{\rho_\alpha(t)} \tag{7}$$

for $i = 1, 2,\ t > 0$, a.s. on Ω_0.

Since

$$|I_i(t)| \leq \frac{1}{\rho_\alpha(t)}\left(\int_{-r}^0 x(t+s)^2\, ds\right)^{1/2} \|\sigma_i\|_2 = \sqrt{1 - a^2(t)}\ \|\sigma_i\|_2,$$

$i = 1, 2,$ a.s. on Ω_0 the variables z_1, z_2, z_3, z_3' in (6) must satisfy

$$|z_1| \leq 1, \ \ z_2 \in \mathbf{R}, \ \ |z_3|^2 \leq (1 - z_1^2)\|\sigma_1\|_2^2, \ \ |z_3'|^2 \leq (1 - z_1^2)\|\sigma_2\|_2^2.$$

Let $\tau_1 := \inf\{t > 0 : \rho_\alpha(t) = 0\}$. Then the local martingale

$$\int_0^{t\wedge\tau_1} R_\alpha(a(u), I_2(u))\, dM(u), \ \ t > 0,$$

is a time-changed (possibly stopped) Brownian motion. Since $|R_\alpha(a(u), I_2(u))| \leq |\nu_2| + \sqrt{\alpha}\|\sigma_2\|_2$ for all $u \in [0, \tau_1)$, a.s., then

$$\lim_{t\to\infty} \frac{1}{t}\int_0^{t\wedge\tau_1} R_\alpha(a(u), I_2(u))\, dM(u) = 0 \qquad \text{a.s.} \tag{8}$$

Divide (5) by t, let $t \to \infty$, to get

$$\lambda((v,\eta),\omega) \leq \limsup_{t\to\infty} \frac{1}{t}\int_0^t Q_\alpha(a(u),b(u),I_1(u))\,du + \limsup_{t\to\infty}\frac{1}{t}\int_0^t \tilde{Q}_\alpha(a(u),I_2(u))\,d\langle M\rangle(u). \tag{9}$$

a.s. on Ω_0, for all $\alpha > 0$.

Wish to develop upper bounds on λ_1 in the following cases.

One-dimensional linear sfde (smooth memory in white-noise term):

$$dx(t) = \{\nu_1 x(t) + \mu_1 x(t-r)\}\,dt + \left\{\int_{-r}^0 x(t+s)\sigma_2(s)\,ds\right\} dW(t), \quad t > 0 \quad (VII)$$

with real constants ν_1, μ_1 and $\sigma_2 \in C^1([-r,0],\mathbf{R})$. It is a special case of (XVII). Hence (VII) is regular with respect to M_2. The process $\int_{-r}^0 x(t+s)\sigma_2(s)\,ds$ has C^1 paths in t. Hence the stochastic differential dW in (VII) may be interpreted in the Itô or Stratonovich sense *without changing the solution* x.

Theorem V.3.

Suppose λ_1 is the top a.s. Lyapunov exponent of (VII). Define the function

$$\theta(\delta,\alpha) := -\delta + \left(\nu_1 + \delta + \frac{1}{2}\alpha\mu_1^2 e^{2\delta r} + \frac{1}{2\alpha}\right) \vee \left(\frac{\alpha}{2}\|\sigma_2\|_2^2 e^{2\delta^+ r}\right)$$

for all $\alpha \in \mathbf{R}^+, \delta \in \mathbf{R}$, where $\delta^+ := \max\{\delta, 0\}$.

Then

$$\lambda_1 \leq \inf\{\theta(\delta,\alpha) : \delta \in \mathbf{R},\ \alpha \in \mathbf{R}^+\}. \tag{10}$$

Proof.

Maximize the integrand on the right-hand-side of (9) (with $M = W$); then use exponential shift by δ to refine the resulting estimate. Then minimize over α, δ ([M-S], II, 1996, pp. 34-35). □

Corollary below shows that the estimate in Theorem V.3 reduces to well-known estimate in deterministic case $\sigma_2 \equiv 0$ (Hale [Ha], pp.17-18).

Corollary V.3.1.

In (VII), suppose $\mu_1 \neq 0$ and let δ_0 be the unique real solution of the transcendental equation

$$\nu_1 + \delta + |\mu_1| e^{\delta r} = 0. \tag{11}$$

Then

$$\lambda_1 \leq -\delta_0 + \frac{1}{2} \frac{\|\sigma_2\|_2^2}{|\mu_1|} e^{|\delta_0| r}. \tag{12}$$

If $\mu_1 = 0$ and $\nu_1 \geq 0$, then $\lambda_1 \leq \frac{1}{2}\big(\nu_1 + \sqrt{\nu_1^2 + \|\sigma_2\|_2^2}\big)$. If $\mu_1 = 0$ and $\nu_1 < 0$, then $\lambda_1 \leq \nu_1 + \frac{1}{2}\|\sigma_2\|_2 e^{-\nu_1 r}$.

Proof.

Suppose $\mu_1 \neq 0$. Denote by $f(\delta)$, $\delta \in \mathbf{R}$, the left-hand-side of (11). Then $f(\delta)$ is an increasing function of δ. f has a unique real zero δ_0. Using (10), we may put $\delta = \delta_0$ and $\alpha = |\mu_1|^{-1} e^{-\delta_0 r}$ in the expression for $\theta(\delta, \alpha)$. This gives (12).

Suppose $\mu_1 = 0$. Put $\delta = (-\nu_1)^+$ in $\theta(\delta, \alpha)$ and minimize the resulting expression over all $\alpha > 0$. This proves the last two assertions of the corollary ([M-S], II, 1996, pp. 35-36). □

Remarks.

(i) Upper bounds for λ_1 in Theorem (V.3) and Corollary V.3.1 agree with corresponding bounds in the deterministic case (for $\mu_1 \geq 0$), but are not optimal when $\mu_1 = 0$ and σ_2 is strictly positive and sufficiently small; cf. Theorem V.1 for small $\|\sigma_2\|_2$.

(ii) *Problem:* What are the asymptotics of λ_1 for small delays $r \downarrow 0$?

Our second example is the stochastic delay equation

$$dx(t) = \{\nu_1 x(t) + \mu_1 x(t-r)\}\, dt + x(t) dM(t), \quad t > 0, \tag{XVIII}$$

where M is the helix local martingale appearing in (XVII) and satisfying the conditions therein. Hence (XVIII) is regular with respect to M_2. Theorem below gives estimate on its top exponent.

Theorem V.4.

In (XVIII) define δ_0 as in Corollary V.3.1. Then the top a.s. Lyapunov exponent λ_1 of (XVIII) satisfies

$$\lambda_1 \leq -\delta_0 + \frac{\beta}{16}. \tag{13}$$

Proof.

Maximize the following functions separately over their appropriate ranges:

$$Q_\alpha(z_1, z_2) := \nu_1 z_1^2 + \sqrt{\alpha}\,\mu_1 z_1 z_2 + \tfrac{1}{2}\frac{z_1^2}{\alpha} - \tfrac{1}{2} z_2^2,$$
$$\tilde{Q}_\alpha(z_1) := (\tfrac{1}{2} - z_1^2) z_1^2, \quad |z_1| \leq 1,\ z_2 \in \mathbf{R}.$$

Then use an exponential shift of the Lyapunov spectrum by an amount δ. Minimize the resulting bound over all α (for fixed δ) and then over all $\delta \in \mathbf{R}$. This minimum is attained if δ solves the transcendental equation (11). Hence the conclusion of the theorem ([M-S], II, 1996, pp. 36-37). □

Remark.

The above estimate for λ_1 is sharp in the deterministic case $\beta = 0$ and $\mu_1 \geq 0$, but is not sharp when $\beta \neq 0$; e.g. $M = W$, one-dimensional standard Brownian motion in the non-delay case ($\mu_1 = 0$). When $M = \nu_2 W$ for a fixed real ν_2, the above bound may be considerably sharpened as in Theorem V.5 below. The sdde in this theorem is a model of dye circulation in the blood stream (cf. Bailey and Williams [B-W], 1996; Lenhart and Travis, 1986).

Theorem V.5. *([M-S], II, 1996).*

For the equation

$$dx(t) = \{\nu_1 x(t) + \mu_1 x(t-r)\}dt + \nu_2 x(t)\, dW(t) \tag{VI}$$

set

$$\phi(\delta) := -\delta + \frac{1}{4\nu_2^2}\left[\left(|\mu_1|e^{\delta r} + \nu_1 + \delta + \frac{1}{2}\nu_2^2\right)^+\right]^2, \tag{14}$$

for $\nu_2 \neq 0$. Then

$$\lambda_1 \leq \inf_{\delta \in \mathbf{R}} \phi(\delta). \tag{15}$$

In particular, if δ_0 is the unique solution of the equation

$$\nu_1 + \delta + |\mu_1|e^{\delta r} + \frac{1}{2}\nu_2^2 = 0, \tag{16}$$

then $\lambda_1 \leq -\delta_0$.

Proof.

Maximize

$$Q_\alpha(z_1, z_2, 0) + \tilde{Q}_\alpha(z_1, 0) = \left(\nu_1 + \frac{1}{2\alpha} + \frac{\nu_2^2}{2}\right) z_1^2 + \sqrt{\alpha}\,\mu_1 z_1 z_2 - \frac{1}{2} z_2^2 - \nu_2^2 z_1^4 \tag{17}$$

for $|z_1| \leq 1, z_2 \in \mathbf{R}$ and then minimize the resulting bound for λ_1 over $\alpha > 0$. Get

$$\lambda_1 \leq \frac{1}{16\nu_2^2}\left[(2\nu_1 + 2|\mu_1| + \nu_2^2)^+\right]^2.$$

The first assertion of the theorem follows from above estimate by applying an exponential shift to (VI). Last assertion of the theorem is obvious ([M-S], II, 1996, pp. 38-39.) □

Problem: Is $\lambda_1 = \inf_{\delta \in \mathbf{R}} \phi(\delta)$?

Remark.

Estimate in Theorem V.5 agrees with the non-delay case $\mu_1 = 0$ whereby $\lambda_1 = \nu_1 - \frac{1}{2}\nu_2^2 = \inf_{\delta \in \mathbf{R}} \phi(\delta)$. Cf. also [AOP], 1986, [B], 1985, and [AKO], 1989.

4. SDDE with Poisson Noise.

Consider the one-dimensional linear delay equation

$$\left.\begin{aligned} dx(t) &= x((t-1)-)\, dN(t) \qquad t > 0 \\ x_0 &= \eta \in D := D([-1,0], \mathbf{R}). \end{aligned}\right\} \tag{V}$$

The process $N(t) \in \mathbf{R}$ is a Poisson process with i.i.d. inter-arrival times $\{T_i\}_{i=1}^{\infty}$ which are exponentially distributed with the same parameter μ. The jumps $\{Y_i\}_{i=1}^{\infty}$ of N are i.i.d. and independent of all the T_i's. Let

$$j(t) := \sup\left\{ j \geq 0 : \sum_{i=1}^{j} T_i \leq t \right\}.$$

Then

$$N(t) = \sum_{i=1}^{j(t)} Y_i.$$

Equation (V) can be solved a.s. in forward steps of lengths 1, using the relation

$$x^{\eta}(t) = \eta(0) + \sum_{i=1}^{j(t)} Y_i x\Big((\sum_{j=1}^{i} T_j - 1)- \Big) \qquad \text{a.s.}$$

Trajectory $\{x_t : t \geq 0\}$ is a Markov process in the state space D (with the supremum norm $\|\cdot\|_{\infty}$). Furthermore, the above relation implies that (V) is regular in D; i.e., it admits a measurable flow $X : \mathbf{R}^+ \times D \times \Omega \to D$ with $X(t, \cdot, \omega) = {}^{\eta}x_t(\cdot, \omega)$, continuous linear in η for all $t \geq 0$ and a.a. $\omega \in \Omega$ (cf. the singular equation (I)).

The a.s. Lyapunov spectrum of (V) may be characterized directly (without appealing to the Oseledec Theorem) by interpolating between the sequence of random times:

$$\tau_0(\omega) := 0,$$

$$\tau_1(\omega) := \inf\left\{ n \geq 1 : \sum_{j=1}^{k} T_j \notin [n-1, n] \quad \text{for all } k \geq 1 \right\},$$

$$\tau_{i+1}(\omega) := \inf\left\{ n > \tau_i(\omega) : \sum_{j=1}^{k} T_j \notin [n-1, n] \text{ for all } k \geq 1 \right\}, \quad i \geq 1.$$

It is easy to see that $\{\tau_1, \tau_2 - \tau_1, \tau_3 - \tau_2, \cdots\}$ are i.i.d. and $E\tau_1 = \mathrm{e}^{\mu}$.

Theorem V.6. *([M-S], II, 1996)*

Let $\xi \in D$ be the constant path $\xi(s) = 1$ for all $s \in [-1, 0]$. Suppose $E \log \|X(\tau_1(\cdot), \xi, \cdot)\|_\infty$ exists (possibly $= +\infty$ or $-\infty$). Then the a.s. Lyapunov spectrum

$$\lambda(\eta) := \lim_{t\to\infty} \frac{1}{t} \log \|X(t, \eta, \omega)\|_\infty, \quad \eta \in D,\ \omega \in \Omega$$

of (V) is $\{-\infty, \lambda_1\}$ where

$$\lambda_1 = e^{-\mu}\, E \log \|X(\tau_1(\cdot), \xi, \cdot)\|_\infty.$$

In fact,

$$\lim_{t\to\infty} \frac{1}{t} \log \|X(t, \eta, \omega)\|_\infty = \begin{cases} \lambda_1 & \eta \notin \operatorname{Ker} X(\tau_1(\omega), \cdot, \omega) \\ -\infty & \eta \in \operatorname{Ker} X(\tau_1(\omega), \cdot, \omega). \end{cases}$$

Proof.

The i.i.d. sequence

$$S_i := \frac{\|(X(\tau_i, \xi, \cdot))\|}{\|(X(\tau_{i-1}, \xi, \cdot))\|} \quad i = 1, 2, \ldots$$

and the LLN give

$$\lim_{n\to\infty}\frac{1}{\tau_n}\log\|(X(\tau_n,\xi,\omega))\| = e^{-\mu}(E\log S_1)$$

for a.a. $\omega\in\Omega$.

Interpolate between the times $\tau_1,\tau_2,\tau_3,\cdots$ to get the continuos limit ([M-S], II, 1996, pp. 27-28). $\square$

VI. MISCELLANY

Geilo, Norway

Saturday, August 3, 1996

14:00-14:50

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VI. MISCELLANY

1. Malliavin Calculus of SFDE's

Objectives.

I) To establish the existence of smooth densities for solutions of $\mathbf{R}^d$-valued sfde's of the form

$$dx(t) = H(t, x_t)\, dt + g(t, x(t-r))\, dW(t). \qquad (XIX)$$

In the above equation, W is an m-dimensional Wiener process, r is a positive time delay, H is a functional $C([-r,0], \mathbf{R}^d) \to \mathbf{R}^d$ and $g : [0,\infty) \times \mathbf{R}^d \to \mathbf{R}^{d\times m}$ is a $d \times m$-matrix-valued function, such that for fixed t the
$d \times d$-matrix $g(t,x)g(t,x)^*$ has degeneracies of polynomial order as x runs on a hypersurface in $\mathbf{R}^d$.

II) Method of proof gives a very general criterion for the hypoellipticity of a class of degenerate parabolic second-order time-dependent differential operators with space-independent principal part.

III) More generally the analysis works when H is replaced by a non-anticipating functional which may depend on the *whole history* of the path ([B-M], Ann. Prob., 1995).

Case $H \equiv 0$ studied by (Bell and Mohammed [B-M], (J.F.A., 1991). Solution $x(t)$ has smooth density wrt Lebesgue measure on $\mathbf{R}^d$ if $g(t,x)g(t,x)^*$ degenerates like $|x|^2$ near 0 (e.g. $g(t,\cdot)$ linear.) Proof uses Malliavin calculus.

Difficulties.

(i) The infinitesimal generator of the trajectory Feller process $\{x_t : t \geq 0\}$ is a highly degenerate second-order differential operator on the state space: its principal part degenerates on a surface of *finite* codimension (Lecture II, Theorem II.3). Hence cannot use existing techniques from pde's.

(ii) Analysis by Malliavin calculus requires derivation of probabilistic lower bounds on the *Malliavin covariance matrix* of the solution x. These bounds are difficult because there is no stochastic flow in the singular case (Lecture III, Theorem III.3). cf. sode case, where stochastic flow is invertible. See work by Kusuoka and Stroock in the uniformly elliptic case ([K-S], I, Taniguchi Sympos. 1982).

(iii) The form of the Malliavin covariance allows *polynomial (finite-type)* rate of degeneracy near a hypersurface, coupled with limited contact of the initial path with the hypersurface. cf. sode case where degeneracies of *infinite type* are compatible with hypellipticity (Bell and Mohammed [B-M], Duke Math. Journal, 1995).

Hypotheses (H).

(i) $W : [0,\infty)\times\Omega\to\mathbf{R}^m$ is standard m-dimensional Wiener process, defined on a complete filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},P)$.

(ii) $g:[0,\infty)\times\mathbf{R}^d\to\mathbf{R}^{d\times m}$ is a continuous map into the space of $d\times m$ matrices, with bounded Fréchet derivatives of all orders in the space variables .

(iii) r is a positive real number, and $\eta:[-r,0]\to\mathbf{R}^d$ is a continuous initial path.

(iv) $H:[0,\infty)\times C\to\mathbf{R}^d$ is a globally bounded continuous map with all Fréchet derivatives of $H(t,\eta)$ wrt η globally bounded in $(t,\eta)\in\mathbf{R}^+\times C$. Think of $H(t,\xi_t)$ as a smooth $\mathbf{R}^d$-valued functional in $\xi\in C([-r,t],\mathbf{R}^d)$. Denote its Fréchet derivative wrt $\xi\in C([-r,t],\mathbf{R}^d)$ by $H_\xi(t,\xi)$. Set

$$\alpha_t := \sup\{\|H_\xi(u,\xi)\| : u\in[0,t],\, \xi\in C([-r,u],\mathbf{R}^d)\}\,, \qquad t>0\,,$$

and

$$\alpha_\infty := \sup\{\|H_\xi(u,\xi)\| : u\in[0,\infty),\, \xi\in C([-r,u],\mathbf{R}^d)\}\,,$$

where $\|H_\xi(u,\xi)\|$ is the operator norm of the partial Fréchet derivative $H_\xi(u,\xi) : C([-r,u],\mathbf{R}^d)\to\mathbf{R}^d$.

Theorem VI.1.

Assume Hypotheses (H) for the sfde (XIX). Suppose there exist positive constants ρ, δ, an integer $p\geq 2$ and a function $\phi:[0,\infty)\times\mathbf{R}^d\to\mathbf{R}$ *satisfying the following conditions*

(i)

$$g(t,x)g(t,x)^*\geq\begin{cases}|\phi(t,x)|^p I, & |\phi(t,x)|<\rho\\ \delta I, & |\phi(t,x)|\geq\rho\end{cases}\tag{1}$$

for $(t, x) \in [0, \infty) \times \mathbf{R}^d$.

(ii) $\phi(t, x)$ *is* C^1 *in* t *and* C^2 *in* x, *with bounded first derivatives in* (t, x) *and bounded second derivatives in* $x \in \mathbf{R}^d$.

(iii) There is a positive constant c *such that*

$$\|\nabla\phi(t, x)\| \geq c > 0 \tag{2}$$

for all $(t, x) \in [0, \infty) \times \mathbf{R}^d$, *with* $|\phi(t, x)| \leq \rho$. *In (2),* ∇ *denotes the gradient operator with respect to the space variable* $x \in \mathbf{R}^d$.

(iv) There is a positive number δ_0 *such that* $\delta_0 < (3\alpha_\infty)^{-1} \wedge r$ *and for every Borel set* $J \subseteq [-r, 0]$ *of Lebesgue measure* δ_0 *the following holds*

$$\int_J \phi(t + r, \eta(t))^2 \, dt > 0. \tag{3}$$

Define $s_0 \in [-r, 0]$ *by*

$$s_0 := \sup\{s \in [-r, 0] : \int_{-r}^{s} \phi(u + r, \eta(u))^2 \, du = 0\}.$$

Then for all $t > s_0 + r$ *the solution* $x(t)$ *of (XIX) is absolutely continuous with respect to d-dimensional Lebesgue measure, and has a* C^∞ *density.*

Remark.

Condition (iv) of Theorem VI.1 may be replaced by the following equivalent condition:

(iv)′ The set $\{s : s \in [-r, o], \ \phi(s, \eta(s)) = 0\}$ has Lebesgue measure less than $(3\alpha_\infty)^{-1} \wedge r$.

Theorem VI.2

In the sfde

$$\left.\begin{aligned} dy(t) &= H(t, y_t)\, dt + F(t) dW(t), \quad t > a \\ y(t) &= x(t), \quad a - r \leq t \leq a, \quad a \geq r \end{aligned}\right\} \tag{XX}$$

suppose that $F : [a, \infty) \to \mathbf{R}^{d\times n}$ *and* $x : [0, a] \to \mathbf{R}^d$ *are continuous. Assume that* H *satisfies regularity hypotheses analogous to (H). For* $t > a$ *let*

$$\alpha'_t := \sup\{\|H_\xi(u,\xi)\| : u \in [0,t],\ \xi \in C\}\,.$$

Suppose there exists $\delta^* < 1/(3\alpha'_t)$ *such that*

$$\int_{t-\delta^*}^{t} \mu_1(s)\, ds > 0, \tag{4}$$

where $\mu_1(s)$, $s \geq a$, *is the smallest eigenvalue of the non-negative definite matrix* $F(s)F(s)^*$. *Then the solution* $y(t)$ *of (XX) has an absolutely continuous distribution with respect to* d*-dimensional Lebesgue measure and has a* C^∞ *density.*

In the special case when $H(t, y) = h(t, y(t))$ in equation (XX) for some Lipschitz function $h : \mathbf{R}^+ \times \mathbf{R}^d \to \mathbf{R}^d$, then y is a (time-inhomogeneous)diffusion process. In this case the proof of Theorem VI.2 gives the following pde result.

Theorem VI.3

For each $t > 0$, let $A(t) = [a_{ij}(t)]_{i,j=1}^{d}$ denote a symmetric non-negative definite $d \times d$-matrix. Let $\mu_2(t)$ be the smallest eigenvalue of $A(t)$. Assume the following:

(i) *The map $t \longmapsto A(t)$ is continuous.*

(ii) *There exists $T > 0$ such that*

$$\int_0^T \mu_2(s)\,ds > 0. \tag{5}$$

(iii) *The functions b_i, $i = 1, \ldots, d$, $c : \mathbf{R}^+ \times \mathbf{R}^d \to \mathbf{R}$ are bounded, jointly continuous in (t, x) and have partial derivatives of all orders in x, all of which are bounded in (t, x). Let $T_0 := \sup\{T > 0 : \int_0^T \mu_2(s)\,ds = 0\}$, and let $L_{t,x}$ denote the differential operator*

$$L_{t,x} := \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(t) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(t,x) \frac{\partial}{\partial x_i} + c(t,x). \tag{6}$$

Then the parabolic equation $\dfrac{\partial u}{\partial t} = L_{t,x} u$ has a fundamental solution $\Gamma(t,x,y)$ defined on $(T_0, \infty) \times \mathbf{R}^{2d}$, which is C^1 in t and C^∞ in (x,y). Furthermore, if the coefficients $a_{ij}(t)$, $b_i(t,x)$, $c(t,x)$, i, $j = 1, \ldots, d$, are C^∞ in (t,x), and

$$\lim_{t \to T_0+} (t - T_0) \log\left\{ \int_{T_0}^{t} \mu_2(s)\,ds \right\} = 0, \tag{7}$$

then $\dfrac{\partial}{\partial t} - L_{t,x}$ is a hypoelliptic operator on $(T_0, \infty) \times \mathbf{R}^d$; (viz. if ϕ is a distribution on $(T_0, \infty) \times \mathbf{R}^d$ such that $\left(\dfrac{\partial}{\partial t} - L_{t,x}\right)\phi$ is C^∞, then ϕ is also C^∞.)

The *mean ellipticity* hypothesis (5) is much weaker than classical *pointwise ellipticity*.

Problem.

Can Theorem VI.3 be proved using existing pde's techniques?

Proof of Theorem VI.1. (Outline)

Objective is to get good probabilistic lower bounds on the Malliavin covariance matrix of the solution $x(t)$ of (XIX). Do this using the following steps:
(cf.[B-M 2], Ann. Prob. 1995, "conditioning argument" using (XX)).
Step 1.

We use piecewise linear approximations of W in (XIX) to compute the Malliavin covariance matrix $C(T)$ of $x(T)$ as

$$C(T) = \int_0^T Z(u) g\big(u, x(u-r)\big) g\big(u, x(u-r)\big)^* Z(u)^* \, du,$$

where the $(d \times d)$-matrix-valued process $Z : [0,T] \times \Omega \to \mathbf{R}^{d\times d}$ satisfies the advanced *anticipating* Stratonovich integral equation

$$\begin{aligned} Z(t) = I &+ \int_{T\wedge(t+r)}^T Z(u) Dg\big(u, x(u-r)\big)(\cdot) \circ dW(u) \\ &+ \int_t^T Z(u) \big[\{H_x(u,x)^*(\cdot)\}'(t)\big]^* \, du, \qquad 0 \le t \le T. \end{aligned}$$

In the above integral equation, $H_x(u,x)$ is the Fréchet partial derivative of the map

$$(u, x) \to H(u, x_u)$$

with respect to $x \in C\big([-r,u], \mathbf{R}^d\big)$. If $W^{1,2}$ is the Cameron Martin subspace of $C\big([-r,u], \mathbf{R}^d\big)$, let $H_x(u,x)^*$ denote the Hilbert-space adjoint of the restriction

$$H_x(u,x)|W^{1,2} : W^{1,2} \to \mathbf{R}^d.$$

We solve the above integral equation as follows.

Start with the terminal condition $Z(T) = I$. On the last delay period $[(T-r)\vee 0, T]$ define Z to be the unique solution of the linear integral equation

$$Z(t) = I + \int_t^T Z(u) \big[\{H_x(u,x)^*(\cdot)\}'(t)\big]^* \, du$$

for a.e. $t \in ((T-r)\vee 0, T)$. When $T > r$, use successive approximations to solve the anticipating integral equation, treating the stochastic integral as a predefined random *forcing term*. This gives a unique solution of the integral equation by successive backward steps of length r. The matrix $Z(t)$ need not be invertible for small t. Compare $Z(t)$ with the analogous process for the diffusion case (sode). In this case $Z(t)$ is invertible for all t and anticipating integrals are not needed.

Step 2.

Since $H_x(u,x)$ is globally bounded in (u,x), then so is $[H_x(u,x)^*(\cdot)]'(t)$ in (u,x,t) ([B-M], Ann. Prob. 1995, Lemma 3.3). Hence can choose a *deterministic* time $t_0 < T$ sufficiently close to T such that almost surely $Z(t)$ is invertible and $\|Z(t)^{-1}\| \leq 2$ for a.e. $t \in (t_0, T]$.([B-M], Ann. Prob. 1995, Lemma 3.4).

Step 3.

The above lower bound on $\|Z(t)\|$ and the representation of $C(T)$ imply that

$$\det C(T) \geq \left[\frac{1}{4}\int_{t_0}^{T} \hat{g}(u, x(u-r))^2\, du\right]^d \qquad \text{a.s.}$$

where

$$\hat{g}(u,v) := \inf\big\{|g(u,v)^*(e)| : e \in \mathbf{R}^d, \quad |e| = 1\big\},$$

for all $u \geq 0,\ v \in \mathbf{R}^d$.

Step 4.

Prove the

Propagation Lemma:

Let $-r < a < b < a + r$. Then the statement

$$P\left(\int_a^b |\phi(u+r, x(u))|^2\, du < \epsilon\right) = o(\epsilon^k)$$

as $\epsilon \to 0+$ *for every* $k \geq 1$,
implies that

$$P\left(\int_{a+r}^{b+r} |\phi(u+r, x(u))|^2 \, du < \epsilon\right) = o(\epsilon^k)$$

as $\epsilon \to 0+$ *for every* $k \geq 1$. (Proof uses Itô's formula, the lower bound on $\|\nabla\phi\|$, the polynomial degeneracy condition and the Kusuoka-Stroock $\epsilon^{1/(18)}$-lemma!).

Step 5.

By successively applying Step 4, we propagate the "limited contact" hypothesis on the initial path η in order to get the estimate:

$$P\left(\int_{t_0}^{T} |\phi(u, x(u-r))|^2 du < \epsilon\right) = o(\epsilon^k)$$

as $\epsilon \to 0+$ for every $k \geq 1$.

Step 6.

Using the polynomial degeneracy hypothesis, Step 5, Jensen's inequality, and Lemma 4.3 of ([B-M], Ann. Prob. 1995), we obtain

$$P\left(\int_{t_0}^{T} \hat{g}\big(u, x(u-r)\big)^2 \, du < \epsilon\right) = o(\epsilon^k)$$

as $\epsilon \to 0+$ for every $k \geq 1$.

Step 7.

Combining steps 3 and 6 gives

$$P\big(\det C(T) < \epsilon\big) = o(\epsilon^k)$$

as $\epsilon \to 0+$ for every $k \geq 1$. This implies that $C(T)^{-1}$ exists a.s. and $\det C(T)^{-1} \in \bigcap_{q=1}^{\infty} L^q(\Omega, \mathbf{R})$. $\square$

2. Back to Square One: Diffusions via SDDE's

Objective is to prove the following existence theorem for classical diffusions using approximations by small delays: (Caratheodory)

Theorem VI.4.(Itô, Gihman-Skorohod,..)

Let $h : \mathbf{R}^d \to \mathbf{R}^d$, $g : \mathbf{R}^d \to \mathbf{R}^{d\times m}$ *be globally Lipschitz, and* W m*-dimensional Brownian motion. Suppose* $x_0 \in \mathbf{R}^d$*. Then the sode*

$$\begin{aligned} dx(t) &= h(x(t))\,dt + g(x(t))\,dW(t), \quad t > 0 \\ x(0) &= x_0 \end{aligned}$$

has an adapted solution with continuous sample paths.

Proof.([B-M], Stochastics, 1989)

For simplicity assume that $h \equiv 0$ and $d = m = 1$.

Fix $0 < T < \infty$. For each integer $k \geq 1$, define

$$\left.\begin{aligned} x^k(t) &= x_0 + \int_0^t g\left(x^k\left(u - \frac{1}{k}\right)\right)\,dW(u), \quad t \geq 0 \\ x^k(t) &= x_0, \quad -\frac{1}{k} \leq t \leq 0 \end{aligned}\right\} \qquad (*)$$

Note that x^k exists, is adapted and continuous.

Step 1.

$x^k : [0,\infty) \to L^2(\Omega, \mathbf{R})$ is $(\frac{1}{2})$-Hölder, with Hölder constant independent of k. (*Exercise:* To prove this observe first that by $(*)$ and the linear growth property of g, there is a positive constant K independent of k and $t \in [0,T]$ such that

$$E \sup_{0\leq u\leq t} (|x^k(t)|^2 + |g(x^k(t)|^2) \leq K$$

for all $k \geq 1$ and all $t \in [0,T]$. Then $E[x^k(t) - x^k(s)]^2 \leq K(t-s)$ for all $t, s \geq 0$.)

Step 2.

For each $t \geq 0$, $x^k(t)$ converges to a limit $x(t)$ in $L^2(\Omega, \mathbf{R}^d)$: Let L be the Lipschitz constant for g.

For $l > k$, we have

$$\begin{aligned}
E[x^l(t) - x^k(t)]^2 &= E\left\{\int_0^t \left[g\left(x^l\left(u - \frac{1}{l}\right)\right) - g\left(x^k\left(u - \frac{1}{k}\right)\right)\right] dW(u)\right\}^2 \\
&\leq L^2 \int_0^t E\left[x^l\left(u - \frac{1}{l}\right) - x^k\left(u - \frac{1}{l}\right)\right]^2 du \\
&\leq 2L^2 \int_0^t E\left[x^l\left(u - \frac{1}{l}\right) - x^k\left(u - \frac{1}{l}\right)\right]^2 du \\
&\qquad + 2L^2 \int_0^t E\left[x^k\left(u - \frac{1}{l}\right) - x^k\left(u - \frac{1}{k}\right)\right]^2 du \\
&\leq 2L^2 \int_{-\frac{1}{l}}^{t-\frac{1}{l}} E[x^l(u) - x^k(u)]^2 \, du + 2KL^2 t\left(\frac{1}{k} - \frac{1}{l}\right) \\
&\leq 2L^2 \int_0^t E[x^l(u) - x^k(u)]^2 \, du + 2t\|a\|_\infty\left(\frac{1}{k} - \frac{1}{l}\right)
\end{aligned}$$

by Step 1. Thus, by Gronwall's lemma

$$E[x^l(t) - x^k(t)]^2 \leq 2TK\left(\frac{1}{k} - \frac{1}{l}\right)e^{2L^2 t}$$

Thus convergence holds. Also

$$E[x(t) - x^k(t)]^2 \leq \frac{2TK}{k} e^{2L^2 t}$$

Step 3.

The process x satisfies original sode:

Simply take limits as $k \to \infty$ in both sides of $(*)$. Then $LHS \to x(t)$ in L^2. x is adapted, since each x^k is. Also

$$
\begin{aligned}
E\Big\{ \int_0^t \Big[g\Big(x^k\Big(u - \frac{1}{k} \Big) \Big) - g(x(u)) \Big] \, dW(u) \Big\}^2 & \\
\le L^2 \int_0^t E\Big[x^k(u - \frac{1}{k}) - x(u) \Big]^2 \, du & \\
\le 2L^2 \int_0^t E\Big[x^k(u - \frac{1}{k}) - x^k(u) \Big]^2 \, du + 2L^2 \int_0^t E[x^k(u) - x(u)]^2 \, du & \\
\le \frac{2L^2 K t}{k} + \frac{2L^2 K T}{k} \int_0^t 2e^{2L^2 u} \, du & \\
\le \frac{2KL^2}{k} \Big[t + \frac{1}{2L^2} \Big(e^{2L^2 t} - 1 \Big) \Big] & \\
\to 0 \text{ as } k \to \infty &
\end{aligned}
$$

Thus

$$
\text{RHS } \to x_0 + \int_0^t g(x(u)) \, dW(u)
$$

i.e. x satisfies the sode.

Since the Itô integral has an a.s. continuous modification, it follows from Doob's inequality that x has such a modification. $\square$

3. Affine SFDE's. A Simple Model of Population Growth

Recall simple population growth model (Lecture I):

$$
dx(t) = \{-\alpha x(t) + \beta x(t-r)\} \, dt + \sigma dW(t), \quad t > 0 \tag{II}
$$

for a large population $x(t)$ with
constant birth rate $= \beta > 0$ (per capita);
constant death rate $= \alpha > 0$ (per capita);
development period $r = (9) > 0$;

migration, white noise, variance σ.

To determine stability, growth rates of population, consider the general affine system :

$$\left.\begin{aligned} dx(t) &= \left\{ \int_{[-r,0]} \mu(ds)x(t+s) \right\} dt + dQ(t), \qquad t > 0 \\ x_0 &= \eta \in D := D([-r,0], \mathbf{R}^d). \end{aligned}\right\} \tag{X}$$

$D :=$ space of all cadlag paths $[-r, 0] \to \mathbf{R}^d$ with $\| \cdot \|_\infty$-norm.

Theorem VI.5.

The Lyapunov spectrum of (X) coincides with the set of all real parts $\{\beta_i : i \geq 1\}$ *of the spectrum of the generator* A *of the homogeneous equation corresponding to* $Q \equiv 0$*, together with possibly* $-\infty$.

We now consider the *hyperbolic case* when $\beta_i \neq 0$ for all $i \geq 1$. In this case, the following result ([M-S1],Theorem 20) establishes the existence of a hyperbolic splitting along a unique stationary solution of (X).

Theorem VI.6. ([M-S], 1990)

Suppose that Q *is cadlag and has stationary increments. Assume that the characteristic equation*

$$\det\left(\lambda I - \int_{[-r,0]} e^{\lambda s}\mu(ds)\right) = 0$$

has no roots on the imaginary axis; i.e., the associated homogeneous equation ($Q \equiv 0$*) has no zero Lyapunov exponents. Suppose also that*

$$\overline{\lim_{t\to\pm\infty}} \frac{1}{|t|} \log|Q(t)| < |\operatorname{Re}\lambda| \qquad a.s.$$

for all characteristic roots λ*. Then there is a unique* D*-valued random variable* η_∞ *such that the trajectory* $\{x_t^{\eta_\infty} : t \geq 0\}$ *of (X) is a* D*-valued stationary process. The random variable* η_∞ *is measurable with respect to the* σ*-algebra generated by* $\{Q(t) : t \in \mathbf{R}\}$*. Furthermore, let* $\beta_1 > \beta_2 > \cdots$ *, be an ordering of the Lyapunov spectrum of (X) and suppose* $\beta_m > 0$*,* $\beta_{m+1} < 0$*. Then there exists a decreasing sequence of finite-codimensional subspaces* $\{E_i : i \geq 1\}$ *of* D *such that*

$$\lim_{t\to\infty} \frac{1}{t} \log\|x_t(\omega)\|_\infty = \beta_i, \qquad i \geq 1$$

if $x_0(\omega) \in \eta_\infty + E_{i-1}\backslash E_i$*,* $1 \leq i \leq m$*, and*

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log\|x_t(\omega)\|_\infty \leq \beta_{m+1}$$

if $x_0(\omega) \in \eta_\infty + E_m$.

Results on the existence of p-*th moment Lyapunov* exponents appear in ([M-S], Stochastics, 1990). Under mild non-degeneracy condition on the stationary solution, one gets the existence of *only* one p-th moment exponent $(= p\beta_1)$ which is independent of all *random* (possibly *anticipating*) initial conditions in D. This result is in agreement with the affine linear finite-dimensional non-delay case $(r = 0)$ ([AOP],[B],[AKO]).

Problem.

Under what conditions on the parameters α, β does (II) have a stationary solution?

Interesting Fact:

The affine hereditary system (X) may be viewed as a *finite-dimensional* stochastic perturbation of the associated *infinitely degenerate* deterministic homogeneous system $(Q \equiv 0)$ with *countably many* Lyapunov exponents. However, these finite-dimensional perturbations provide noise that is generically rich enough to account for a *single* $p-$th moment Lyapunov exponent in the affine stochastic system (X).

4. Random Delays

See [Mo], Pitman Books, 1984, pp. 167-186. Delays are allowed to be random (independently) of the noise and essentially bounded. Markov property fails, but get a measure-valued process with random *Markov* transition measures on state space.

5. Infinite Delays. Stationary Solutions

Pioneering work of Itô and Nisio, [I-N], J. Math. Kyoto University, 1964, pp. 1-75. Results summarized in [Mo], 1984, pp. 230-233.

6. Summary

Main themes are:

SMOOTH HISTORY $\iff$ REGULARITY OF SFDE

FINITE HISTORY $\Longrightarrow$ LOCAL COMPACTNESS OF SEMI-FLOW $\Longrightarrow$ DISCRETE LYAPUNOV SPECTRUM $\Longrightarrow$ HYPERBOLICITY AND STABLE MANIFOLDS

HELIX NOISE $\Longrightarrow$ NON-IVERTIBLE MULTIPLICATIVE COCYCLE ON HILBERT SPACE

DISCRETE FINITE HISTORY $\Longrightarrow$ SINGULAR SYSTEM $\Longrightarrow$ NONEXISTENCE OF FLOW

STOCHASTIC DIFFERENTIAL SYSTEMS WITH MEMORY

THEORY, EXAMPLES AND APPLICATIONS

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